THÈSE DE DOCTORAT

Présentée par

Xavier FRIEDERICH

En vue de l'obtention du grade de docteur de l'Université de Strasbourg

Spécialité *Mathématiques*

✶ ✶ ✶

PROPRIÉTÉS QUALITATIVES DES MULTI-SOLITONS

Soutenue le 11 juin 2021

Devant le jury composé de

M. Raphaël CÔTE	Directeur de thèse
M. Rémi CARLES	Rapporteur
M. Stephen GUSTAFSON	Rapporteur
Mme Nalini ANANTHARAMAN	Examinatrice
Mme Valeria BANICA	Examinatrice
M. Frank MERLE	Examinateur

✶✶✶

UNIVERSITÉ DE STRASBOURG
INSTITUT DE RECHERCHE MATHÉMATIQUE AVANCÉE UMR 7501

ECOLE DOCTORALE ED 269 - MSII

Remerciements

Je tiens à exprimer toute ma reconnaissance envers les personnes qui m'ont permis de mener à bien mon parcours doctoral.

Je remercie tout particulièrement mon directeur de thèse, Raphaël Côte, sans qui mes travaux n'auraient pas vu le jour. Ses encouragements et conseils, sa disponibilité, ses intuitions brillantes et sa confiance dans l'évolution de ma thèse ont constitué des richesses colossales à mon égard.

Je remercie chaleureusement les rapporteurs de cette thèse, Rémi Carles et Stephen Gustafson, pour leur lecture attentive du manuscrit et leurs observations ainsi que les autres membres du jury qui m'ont fait l'honneur d'accepter d'évaluer ce travail.

Cette thèse s'accompagne de l'achèvement de mes études à l'Université de Strasbourg. Si je suis arrivé à ce stade, c'est indéniablement grâce à la contribution des différents enseignants qui ont croisé mon parcours d'études. Aussi, j'aimerais témoigner plus que jamais de ma satisfaction d'avoir bénéficié de savoirs inestimables, ne serait-ce que d'une infime partie, et de visions très variées, éclairantes, des mathématiques.

Je suis en outre très heureux d'avoir eu moi-même la possibilité d'enseigner les mathématiques à l'Université et je ne peux que regretter que les séances de cours et de travaux dirigés avec les étudiants, même à distance, se soient déroulées si rapidement. J'ai ici une pensée envers les collègues et le personnel administratif et technique qui ont œuvré au bon déroulement de ces enseignements.

Cette thèse n'aurait sans doute pas eu le même goût sans les échanges avec les autres doctorants en mathématiques, certains parmi lesquels m'étaient connus de plus longue date. Je suis convaincu que plusieurs de nos chemins se recroiseront à l'avenir.

Je souhaite enfin saluer mes proches qui m'ont soutenu tout au long de ma formation doctorale, à la fois de près et de loin.

Xavier Friederich

Résumé

Cette thèse est consacrée aux propriétés qualitatives des multi-solitons de trois équations aux dérivées partielles non-linéaires dispersives focalisantes.

Les premiers résultats nouveaux concernent l'équation de Schrödinger non-linéaire (NLS). Nous construisons des multi-solitons réguliers, à valeurs dans des espaces de Sobolev H^s, où $s > 1$ est directement relié à la régularité de la non-linéarité. Nous démontrons également un résultat d'unicité sous condition pour les multi-solitons de (NLS) dans les cas stable et L^2-critique.

Par ailleurs, nous énonçons une propriété de Liouville au voisinage des multi-solitons de l'équation de Korteweg-de Vries généralisée (gKdV). Il s'agit d'un résultat de rigidité qui repose sur le concept de *non-dispersion*. En particulier dans les cas intégrables, ce concept permet de caractériser les multi-solitons. Dans une autre mesure, nous étudions le comportement ponctuel des multi-solitons de (gKdV).

Enfin, nous nous intéressons à l'équation de Klein-Gordon non-linéaire (NLKG). Pour cette dernière, nous construisons une famille à N paramètres de N-solitons à décroissance exponentielle en temps. En ce qui concerne la question de l'unicité, nous classifions les multi-solitons dans une certaine classe à décroissance polynomiale et obtenons la classification générale dans le cas où l'on considère une seule onde solitaire.

Abstract

This thesis is devoted to the qualitative properties of the multi-solitons of three nonlinear focusing dispersive partial differential equations.

The first new results concern the nonlinear Schrödinger equation (NLS). We construct smooth multi-solitons taking values in Sobolev spaces H^s, where $s > 1$ is directly related to the regularity of the nonlinearity. We also prove conditional uniqueness for the multi-solitons of (NLS) in the stable and L^2-critical cases.

Moreover we state a Liouville property in the neighborhood of the multi-solitons of the generalized Korteweg-de Vries equation (gKdV). It consists in a rigidity result which relies on the concept of *non dispersion*. In particular in the integrable cases, this concept allows us to characterize the multi-solitons. From another perspective, we study the pointwise behavior of the multi-solitons of (gKdV).

Lastly we are interested in the nonlinear Klein-Gordon equation (NLKG). For the latter, we construct an N-parameter family of N-solitons with exponential decrease in time. Regarding the question of uniqueness, we classify the multi-solitons in a certain class with algebraic decay and we obtain the general classification in the case where only one solitary wave is considered.

Contents

Chapter 1

Introduction

1.1 Solitons, multi-solitons et théorie non-linéaire des équations aux dérivées partielles dispersives

1.1.1 Les solitons dans le cadre des équations de Korteweg-de Vries généralisées et de Schrödinger non-linéaires

D'un point de vue historique, l'équation de Korteweg-de Vries (non-linéaire)

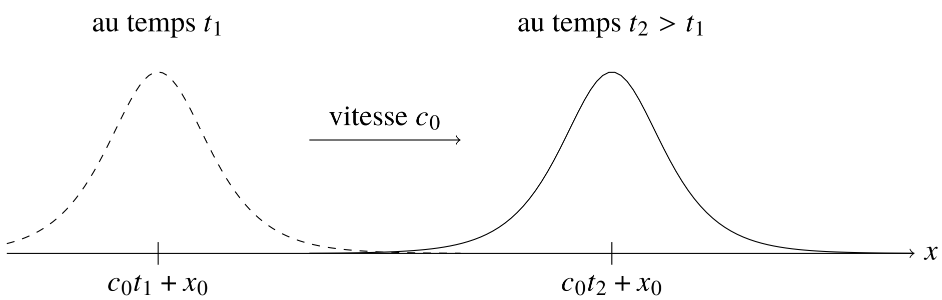
$$\partial_t u = -\partial_x \left(\partial_x^2 u + u^2\right) \tag{KdV}$$

a vu le jour au dix-neuvième siècle avec l'observation, dans des eaux peu profondes, de vagues qui prennent la forme d'ondes solitaires se propageant au cours du temps sans se déformer.

La structure de (KdV) présente l'intérêt d'admettre une famille complète de solutions, indexée par deux paramètres, la vitesse $c_0 > 0$ et la position $x_0 \in \mathbb{R}$, qui correspondent aux ondes solitaires précédentes. Les éléments R_{c_0,x_0} de cette famille sont appelés *solitons* et prennent la forme explicite suivante :

$$R_{c_0,x_0}(t,x) = \frac{3c_0}{2\cosh^2\left(\frac{\sqrt{c_0}}{2}(x - c_0 t - x_0)\right)}.$$

Le soliton R_{c_0,x_0} est donc une solution, qui à chaque temps t, a un profil de courbe en cloche centrée en $c_0 t + x_0$ et qui se déplace au cours du temps vers la droite à vitesse c_0, sans changer de forme ni d'amplitude.



Dans la même optique, nous considérons les équations de Korteweg-de Vries généralisées suivantes qui admettent encore un nombre important d'applications à la physique [55, 89] :

$$\partial_t u = -\partial_x \left(\partial_x^2 u + u^p\right) \tag{gKdV}$$

avec $p > 1$ et où u est une fonction des variables unidimensionnelles t et x et à valeurs réelles.

Un cadre naturel d'étude et d'appartenance des solutions de (gKdV) est l'espace de Sobolev $H^1(\mathbb{R})$. Le problème de Cauchy associé à (gKdV) est en effet localement bien posé dans $H^1(\mathbb{R})$ selon Kenig, Ponce et Vega [50].

Il est remarquable d'observer que (gKdV) admet une famille de solitons, encore indexée par $\mathbb{R}_+^* \times \mathbb{R}$ et qui généralise la famille de solitons de (KdV) de la manière suivante : notant Q l'unique solution positive (à translation près) dans $H^1(\mathbb{R})$ [2, 54] de l'équation elliptique stationnaire associée à (gKdV)

$$Q'' + Q^p = Q,$$

qui admet l'expression

$$Q(x) = \left(\frac{p+1}{2\cosh^2\left(\frac{p-1}{2}x\right)}\right)^{\frac{1}{p-1}},$$

et $Q_{c_0}(x) := c_0^{\frac{1}{p-1}} Q(\sqrt{c_0}x)$, les éléments R_{c_0,x_0} définis par

$$R_{c_0,x_0}(t,x) = Q_{c_0}(x - c_0 t - x_0)$$

sont des solutions globales de (gKdV) qui sont les solitons de (gKdV).

Les solitons sont des objets très particuliers, dont l'existence est liée à la propriété focalisante de (gKdV), due au signe de la non-linéarité u^p : il y a un certain effet compensatoire entre ce terme non-linéaire et la partie linéaire de l'équation fondamentale qui est dispersive en raison de la dérivée seconde.

Nous rappelons aussi que les quantités suivantes sont conservées (au moins formellement) pour les solutions de (gKdV) :

- (la masse ou la norme L^2) $\displaystyle\int_{\mathbb{R}} u^2(t,x)\, dx$
- (l'énergie) $\displaystyle\int_{\mathbb{R}} \left\{\frac{1}{2}u_x^2 - \frac{1}{p+1}u^{p+1}\right\}(t,x)\, dx.$

La norme H^1 n'est en revanche pas constante au cours du temps en général. En outre, la norme $\dot{H}^{\sigma(p)}$, où $\sigma(p) := \frac{1}{2} - \frac{2}{p-1}$ est conservée par l'invariance d'échelle de (gKdV), selon laquelle pour tout $\lambda > 0$, $(t,x) \mapsto \lambda^{\frac{2}{p-1}} u\left(\lambda^3 t, \lambda x\right)$ est solution de (gKdV) dès lors que u l'est. Il s'agit de noter que la dynamique des solutions est liée intimement au signe de $\sigma(p)$, ce qui donne lieu à la distinction des cas :

- L^2-sous-critique, correspondant à $\sigma(p) < 0$ (ou $1 < p < 5$), où toutes les solutions H^1 de (gKdV) sont globales en temps et uniformément bornées dans H^1
- L^2-critique, correspondant à $\sigma(p) = 0$ (ou $p = 5$), où il existe des solutions qui explosent en temps fini [68, 69, 77–79]
- L^2-surcritique, correspondant à $\sigma(p) > 0$ (ou $p > 5$), où il existe des solutions qui explosent en temps fini au moins pour tout $p \in (5, p^*)$ pour un certain réel $p^* > 5$ [56].

Il s'avère que les critères connus d'existence de solutions qui explosent en temps fini lorsque $p \geq 5$ reposent sur l'existence et les propriétés des solitons.

Notons enfin que les cas particuliers de (gKdV) qui correspondent à $p = 2$ (KdV) et $p = 3$ (mKdV) (ou équation de Korteweg-de Vries modifiée) sont des équations complètement intégrables : elles admettent une infinité de lois de conservation [58, 89].

Un autre exemple important d'équation aux dérivées partielles dispersive non-linéaire qui admet des solitons est fourni par l'équation de Schrödinger :

$$\partial_t u = i\left(\Delta u + |u|^{p-1}u\right), \tag{NLS}$$

où $1 < p < 1 + \frac{4}{(d-2)_+}$, (t, x) appartient à $\mathbb{R} \times \mathbb{R}^d$ et u est à valeurs dans $\mathbb{C}$.

Dans le contexte de (NLS), le problème de Cauchy est encore localement bien posé dans $H^1(\mathbb{R}^d)$ selon un résultat de Ginibre et Velo [34]. De plus, les quantités suivantes sont conservées par les solutions H^1 de (NLS) :

- la masse ou norme L^2 $\displaystyle\int_{\mathbb{R}^d} |u(t,x)|^2\,dx$
- l'énergie $\displaystyle\int_{\mathbb{R}^d} \left(\frac{1}{2}|\nabla u(t,x)|^2 - \frac{1}{p+1}|u(t,x)|^{p+1}\right)\,dx$
- le moment $\displaystyle\mathrm{Im}\int_{\mathbb{R}^d} \nabla u(t,x)\overline{u}(t,x)\,dx$.

L'équation adimensionnée (NLS) que nous considérons est aussi au cœur d'un certain nombre de modélisations de phénomènes physiques et tout particulièrement optiques. Pour ne citer qu'un exemple, l'enveloppe complexe E du champ électrique associé à un faisceau d'onde de lumière stationnaire qui se propage le long de l'axe des z dans un milieu d'indice de réfraction non-linéaire $n = n_0 + \delta E^2$ avec $\delta > 0$ est bien décrite par l'équation

$$2ik\frac{\partial E}{\partial z} + \frac{\partial^2 E}{\partial x^2} = -k^2\frac{\delta}{n_0}|E|^2 E, \tag{1.1}$$

où k est le nombre d'onde associé [49]. D'un point de vue mathématique, il revient bien sûr au même d'étudier cette dernière équation ou d'étudier (NLS) cubique (c'est-à-dire avec $p = 3$).

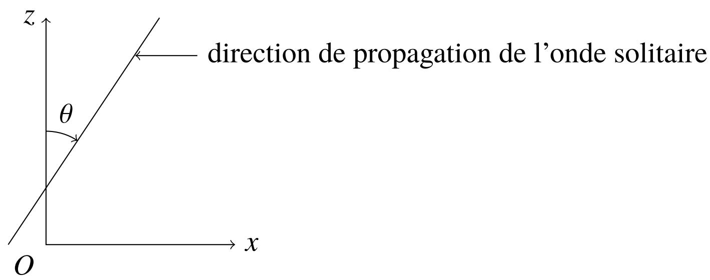
En dimension 1, (NLS) cubique se réécrit

$$\frac{\partial u}{\partial t} = i\left(\frac{\partial^2 u}{\partial x^2} + |u|^2 u\right)$$

et possède des solitons qui sont des solutions explicites particulières ayant une expression de la forme

$$Q(t,x) := \sqrt{2}\frac{e^{-4i(\xi^2-\eta^2)t-2i\xi x+i\phi}}{\cosh(2\eta(x-x_0)+8\eta\xi t)},$$

où $\xi \in \mathbb{R}$, $\eta > 0$, $x_0 \in \mathbb{R}$ et $\phi \in \mathbb{R}$. Les solitons précédents sont ainsi caractérisés par les quatre paramètres de taille η, de vitesse ξ, de position x_0 et de phase ϕ. Dans le contexte physique de l'équation (1.1), la variable z remplace t et le soliton précédent $Q(z,x)$ correspond à une onde solitaire qui se propage sans se déformer, dans la direction donnée par un angle $\theta = -\arctan(4\xi)$ avec l'axe des z.



En dimension supérieure, on sait encore démontrer, pour tout $\omega > 0$, l'existence (travaux de Berestycki et Lions [2]) et l'unicité (travaux de Kwong [54]) de solutions fondamentales Q_ω qui vérifient le problème stationnaire

$$\Delta Q + Q^p = \omega Q,$$

mais on ne sait pas expliciter de telles solutions. Par conséquent, nous n'avons pas de formule explicite pour les solitons construits à partir d'une telle solution fondamentale et des propriétés d'invariance de (NLS) par translation, par rotation et par transformation galiléenne.

1.1.2 Pourquoi les multi-solitons ?

Il convient de souligner que les solitons sont des objets centraux dans la compréhension et l'analyse du comportement des solutions des équations aux dérivées partielles dispersives non-linéaires focalisantes.

Nous avons déjà vu par exemple que les solitons permettent de dégager des conditions pour assurer l'existence de solutions de (gKdV) qui explosent en temps fini dans le cas L^2-critique [68]. Mais ces solutions très particulières ont aussi un rôle essentiel dans l'étude de la dynamique de solutions générales, définies en temps long.

Considérons par exemple la question de la stabilité des solitons qui est un pas naturel pour comprendre la dynamique des équations dispersives non-linéaires focalisantes. Il s'agit en fait d'étudier le comportement de solutions qui sont, à l'instant initial, proches d'un soliton. La notion de *stabilité orbitale* que l'on définit dès à présent est appropriée au contexte des ondes solitaires.

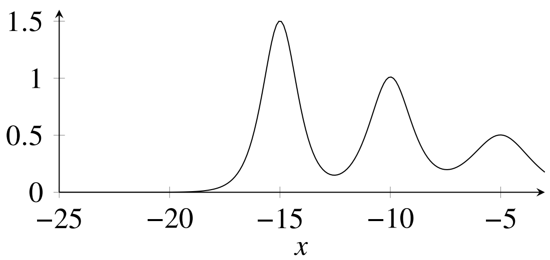
Definition 1.1. *Soit $u : [0,T) \to H^1$ une solution d'une équation aux dérivées partielles (E) et soit G un groupe qui agit sur l'ensemble $\{u(t) \mid t \in [0,T)\}$. On dit que u est G*-orbitalement stable *(ou orbitalement stable s'il n'y a pas d'ambiguité) si pour tout* $\epsilon > 0$*, il existe* $\delta > 0$ *tel que pour toute solution* v *de* (E) *définie sur* $[0, T)$ *et à valeurs dans* H^1*,*

$$\|v(0) - u(0)\|_{H^1} \leq \delta \quad \Rightarrow \quad \forall\, t \in [0, T),\ \inf_{g \in G} \|v(t) - g \cdot u(t)\|_{H^1} \leq \epsilon.$$

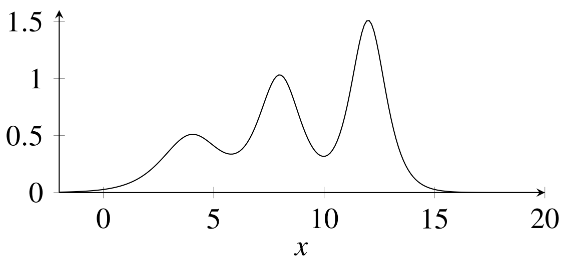
Mais surtout, l'introduction du concept de soliton et son étude tirent essentiellement leur motivation de la conjecture de résolution en solitons, selon laquelle toute solution générique d'une équation aux dérivées partielles non-linéaire dispersive focalisante se décompose en temps long comme la somme de solitons (ou, plus largement, de solutions localisées) et d'un terme dispersif linéaire.

Rappelons qu'à l'origine de cette conjecture, un phénomène d'interaction et de collision élastique a été observé et étudié numériquement par Zabusky et Kruskal [112] : des solutions particulières de (KdV) dont le profil correspond à celui de la superposition de plusieurs ondes solitaires de vitesses deux à deux distinctes avant collision conservent un profil similaire après collision. Les caractéristiques physiques comme l'amplitude des « pulses » des ondes solitaires observées restent les mêmes, seul l'ordre d'apparition en espace de ces pulses est inversé.

Profil typique de la solution observé avant collision



Profil typique de la solution observé après collision



La méthode de scattering inverse développée par Gardner, Greene, Kruskal et Miura [32] confirme et justifie l'observation de Zabusky et Kruskal d'un point de vue théorique. Les résultats obtenus par cette méthode vont pleinement dans le sens de la conjecture de résolution en solitons.

Dans le cas de l'équation intégrable (KdV), un résultat de résolution en solitons a été notamment élaboré par Eckhaus et Schuur [28] puis raffiné par Schuur [101] à partir de la méthode précédente. Il concerne les solutions de (KdV) génériques, au sens où la donnée initiale correspondante est suffisamment régulière et à décroissance suffisamment rapide.

Théorème 1.2 (Eckhaus et Schuur [28]; Schuur [101])**.** *Soit $p = 2$ et soit $u_0 \in \mathscr{C}^4(\mathbb{R})$ une fonction telle qu'il existe $C_0 > 0$ tel que pour tout $k = 0, \dots, 4$ et $x \in \mathbb{R}$,*

$$\left| \frac{\partial^k u_0(x)}{\partial x^k} \right| \leq C_0 |x|^{-11}. \tag{1.2}$$

Soit u la solution globale de (KdV) ayant pour condition initiale u_0. Alors il existe une solution u_d qui est un multi-soliton ou la solution nulle telle que pour tout $\beta > 0$, il existe $K \geq 0$ tel que pour tout $t > 0$

$$\|u(t) - u_d(t)\|_{L^\infty(x>\beta t)} + \|u(t) - u_d(t)\|_{L^2(x>\beta t)} \leq K t^{-\frac{1}{3}}. \tag{1.3}$$

Un autre résultat de résolution en solitons et breathers a été donné pour des solutions « génériques » de (mKdV) par Schuur [101] et Chen et Liu [8] (voir Chapitre 3).

En vue d'éclairer d'une certaine manière l'étude de la décomposition en temps long des solutions à l'aide des solitons, nous introduisons le concept de multi-soliton.

Définition 1.3. *Considérons $N \geq 1$ solitons $R_1, \dots, R_N$ d'une équation aux dérivées partielles non-linéaire (E) ayant des vitesses deux à deux distinctes. Un multi-soliton en $+\infty$ (resp. en $-\infty$) associé aux R_i, $i = 1, \dots, N$, est une solution u de (E) définie au voisinage de $+\infty$ (resp. $-\infty$), à valeurs dans $H^1(\mathbb{R}^d)$ et telle que*

$$\left\| u(t) - \sum_{i=1}^N R_i(t) \right\|_{H^1} \longrightarrow 0, \qquad \text{quand } t \longrightarrow +\infty \text{ (resp. quand } t \longrightarrow -\infty). \tag{1.4}$$

Typiquement pour (gKdV), nous prenons pour $i = 1, \dots, N$,

$$R_i(t,x) := Q_{c_i}(x - c_i t - x_i),$$

où $0 < c_1 < \dots < c_N$ et $x_1, \dots, x_N \in \mathbb{R}$. Pour (NLS), il est classique de travailler avec K solitons (au lieu de N solitons), et de considérer pour tout $k = 1, \dots, K$,

$$R_k(t,x) := Q_{\omega_k}(x - x_k^0 - v_k t) e^{i\left(\frac{1}{2} v_k \cdot x + \left(\omega_k - \frac{|v_k|^2}{4}\right) t + \gamma_k\right)},$$

où les différents paramètres qui interviennent dans l'écriture sont $\omega_k > 0$, $\gamma_k \in \mathbb{R}$, $x_k^0 \in \mathbb{R}^d$ et $v_k \in \mathbb{R}^d$ tels que pour tout $k \neq k'$, $v_k \neq v_{k'}$.

Se posent d'emblée un certain nombre de questions en ce qui concerne les multi-solitons. D'abord, ces objets existent-ils toujours ? Peut-on les construire, les caractériser, en donner des formules explicites ? Ces solutions particulières sont-elles régulières, uniques, stables... ? Peut-on en préciser le comportement en temps, en espace ?

Notre objectif est de répondre, dans la mesure du possible, aux questions précédentes. En vérité, on retrouve les multi-solitons dans différents contextes. A présent, on sait démontrer leur existence via une preuve constructive. En revanche, beaucoup d'autres propriétés qualitatives des multi-solitons sont encore assez peu comprises. Il convient de noter toutefois que le cadre des équations de Korteweg-de Vries généralisées (dont la structure particulière permet aux méthodes usuelles d'analyse non-linéaire d'aboutir) offre une bonne compréhension des multi-solitons ; la classification complète, dans chacun des cas L^2-sous-critique, critique et sur-critique a été dressée. Cela dit, même dans le contexte de (gKdV), certaines questions demeurent sans réponse à ce jour. Aussi, nous souhaitons dégager de nouveaux résultats concernant les multi-solitons de (gKdV) et (NLS), ainsi que pour les équations de Klein-Gordon non-linéaires (NLKG) que nous introduisons dans la partie 1.4.

D'autres problèmes qui dépassent le cadre de cette thèse restent encore ouverts aujourd'hui. Par exemple, il serait intéressant de pouvoir décrire le comportement en $-\infty$ des multi-solitons en $+\infty$ qui sont définis pour tout temps $t \in \mathbb{R}$. Une approche naturelle serait évidemment de relier la décomposition en somme de solitons en $+\infty$ avec une éventuelle autre décomposition en $-\infty$ pour de telles solutions, et pour cela, d'essayer de mieux comprendre le phénomène de collision des différentes ondes solitaires en jeu.

En appliquant la méthode de scattering inverse à (KdV), on obtient à ce propos l'existence de solutions qui sont à la fois des multi-solitons en $+\infty$ et en $-\infty$: précisément, étant donnés N solitons $R_1, \ldots, R_N$ associés à des paramètres de vitesses $0 < c_1 < \cdots < c_N$ et de positions $x_1, \ldots, x_N$, il existe une solution u telle que

$$\left\| u(t) - \sum_{i=1}^{N} R_i(t) \right\|_{H^1} \to 0, \quad \text{quand } t \to +\infty$$

et

$$\left\| u(t) - \sum_{i=1}^{N} R_i(t)(\cdot - \delta_i) \right\|_{H^1} \to 0, \quad \text{quand } t \to -\infty,$$

où $\delta_i \in \mathbb{R}$ est quantifiable à l'aide des vitesses c_i.
En outre, cette solution u est explicite et peut s'écrire (voir par exemple [32], [89] ou [39]) :

$$u = 6\frac{\partial^2}{\partial x^2} \ln \det M,$$

où $M(t,x)$ est la matrice carrée de taille N dont l'élément générique $M_{(i,j)}(t,x)$ est égal à

$$\delta_{i,j} + 2\frac{(c_i c_j)^{\frac{1}{4}}}{\sqrt{c_i} + \sqrt{c_j}} e^{\frac{1}{2}\left(\sqrt{c_i}(x - c_i t) + x_i + \sqrt{c_j}(x - c_j t) + x_j\right)}.$$

Des formules analogues peuvent être obtenues pour (mKdV) [101, chapitre 6] et (NLS) en dimension 1 avec la non-linéarité $|u|^2 u$.

En utilisant la méthode de scattering inverse, Zakharov et Shabat [113] construisent un multi-soliton pour (NLS) en dimension $d = 1$. Là encore, le multi-soliton obtenu se découple en des solitons de mêmes fréquences et mêmes vitesses lorsque $t \to \pm\infty$; les seuls paramètres modifiés sont la position et la phase.

Hormis dans les cas intégrables précédents où la collision entre les différents solitons est élastique (au sens où les vitesses des différentes ondes solitaires en lesquelles se découplent les multi-solitons restent inchangées), le comportement asymptotique en $-\infty$ d'un multi-soliton en $+\infty$ qui est de plus global n'est pas encore bien connu. Les estimées asymptotiques dont on dispose en $+\infty$

pour les multi-solitons définis au voisinage de $+\infty$ ne permettent guère de préciser le phénomène de collision qui se produit.

Cependant, d'après un résultat de collision rigoureux dû à Martel et Merle [73–75] qui fournit des estimées très précises des différents paramètres impliqués, nous savons qu'un 2-soliton en $+\infty$ de (gKdV) avec $p = 4$ ne peut pas se comporter comme une somme de solitons en $-\infty$. Notons en outre que Muñoz [90] a généralisé ce premier résultat de collision inélastique à des non-linéarités sous-critiques plus générales pour (gKdV).

Bien que nous n'en faisions pas l'étude ici, signalons aussi, outre les multi-solitons, l'existence d'autres solutions particulières en lien avec la conjecture de résolution en solitons. Aussi, Côte [14] a démontré, pour des fonctions v suffisamment décroissantes, l'existence de solutions u de (gKdV) définies en temps long et telles que

$$\|u(t) - S(t)v\|_{H^1} \to 0, \quad \text{lorsque } t \to +\infty,$$

où $\{S(t)\}_t$ est le groupe linéaire d'Airy : de telles solutions u dispersent donc complètement, en se comportant comme des solutions linéaires en temps long. Par ailleurs, Côte a également construit, à nouveau pour une certaine classe de fonctions v, des solutions u de (gKdV) pour $p = 4$ [13] et $p = 5$ [15] qui se comportent asymptotiquement comme une somme de solitons et d'un terme dispersif linéaire :

$$\left\| u(t) - \sum_{i=1}^{N} R_i(t) - S(t)v \right\|_{H^1} \to 0, \quad \text{lorsque } t \to +\infty.$$

Remarquons que la solution précédente possède en quelque sorte une dynamique plus complexe que les multi-solitons puisqu'elle se découple en temps long en une partie non-linéaire décrite par la somme des solitons qui se propagent vers la droite et aussi en une partie linéaire qui se propage vers la gauche.

1.1.3 Des outils d'analyse non-linéaire adaptés à l'étude des multi-solitons

Les outils classiques d'analyse linéaire comme les transformations de Fourier ou de Laplace tombent rapidement en défaut lorsqu'il s'agit d'étudier des équations non-linéaires comme (gKdV) et (NLS). Il est usuel de recourir alors à des méthodes plus appropriées de point fixe, de compacité, d'énergie, et autres fondées sur l'étude de l'évolution de certaines quantités dans le temps.
Mais il est souvent utile, voire indispensable, de considérer un cadre linéaire proche du problème non-linéaire qui fait l'objet d'étude. L'intérêt est de pouvoir manipuler des opérateurs linéaires et, si possible, de considérer leur spectre qui fournit des renseignements précieux quant à la dynamique générale des solutions de l'équation étudiée.
Aussi, lorsqu'on travaille plus particulièrement avec des solitons, il s'avère utile d'introduire les opérateurs linéarisés au voisinage de chacun de ces solitons.

Par exemple, pour tout $c > 0$, l'équation linéarisée de (gKdV) autour de $Q_c + v$ s'écrit $\partial_t v = \partial_x (L_c v)$ où L_c est l'opérateur linéaire $H^1(\mathbb{R}) \to H^1(\mathbb{R})$ d'expression

$$L_c v = -\partial_x^2 v + cv - pQ_c^{p-1} v.$$

L'article de Pego et Weinstein [96] s'attache à la théorie spectrale de l'opérateur $\partial_x L_c$; celle-ci est le point de départ pour démontrer que L_c est strictement coercif en restriction à des sous-espaces bien choisis de $H^1(\mathbb{R})$. Ces sous-espaces dépendent du régime L^2-sous-critique, critique ou surcritique considéré, autrement dit de la valeur de p.
Les cas L^2-sous-critique et critique sont traités dans l'article [109] de Weinstein. Dans le cas L^2-surcritique, l'existence de vecteurs propres Z^+ et Z^- associés aux valeurs propres respectives $e_0 > 0$ et $-e_0$ [96] suppose une propriété de coercivité différente, démontrée par Côte, Martel et Merle [19] (à partir des idées de Duyckaerts et Merle [24]) ; les directions instables Z^+ et Z^- doivent notamment être contrôlées.
Ces propriétés de coercivité remarquables font l'objet de la proposition suivante.

Proposition 1.4. *Soit $c > 0$.*
Si $1 < p < 5$, alors il existe $\lambda > 0$ tel que pour tout $v \in H^1(\mathbb{R})$,

$$\text{si} \quad \int_{\mathbb{R}} vQ_c\, dx = \int_{\mathbb{R}} vQ_c'\, dx = 0, \quad \text{alors} \quad \int_{\mathbb{R}} L_c(v)v\, dx \geq \lambda \|v\|_{H^1}^2.$$

Si $p = 5$, alors il existe $\lambda > 0$ tel que pour tout $v \in H^1(\mathbb{R})$,

$$\text{si} \quad \int_{\mathbb{R}} vQ_c^3\, dx = \int_{\mathbb{R}} vQ_c'\, dx = 0, \quad \text{alors} \quad \int_{\mathbb{R}} L_c(v)v\, dx \geq \lambda \|v\|_{H^1}^2.$$

Si $p > 5$, alors il existe $\lambda > 0$ tel que pour tout $v \in H^1(\mathbb{R})$,

$$\text{si} \quad \int_{\mathbb{R}} vZ^+\, dx = \int_{\mathbb{R}} vZ^-\, dx = \int_{\mathbb{R}} vQ_c'\, dx = 0 \quad \text{alors} \quad \int_{\mathbb{R}} L_c(v)v\, dx \geq \lambda \|v\|_{H^1}^2.$$

Pour (NLS), considérant $\omega > 0$, l'équation linéarisée de (NLS) autour de $e^{i\omega t}(Q_\omega + w)$ s'écrit $\partial_t w = i\mathscr{L}_\omega w$ où

$$\begin{array}{rccl} \mathscr{L}_\omega : & H^1(\mathbb{R}^d, \mathbb{C}) & \to & H^1(\mathbb{R}^d, \mathbb{C}) \\ & w = w_1 + iw_2 & \mapsto & -\Delta w + \omega w - \left(Q_\omega^{p-1} w + (p-1) Q_\omega^{p-1} w_1 \right). \end{array}$$

L'opérateur $\mathscr{L}_\omega$ se décompose de façon immédiate à l'aide de deux opérateurs $L_{+,\omega}, L_{-,\omega}$: $H^1(\mathbb{R}^d, \mathbb{R}) \to H^1(\mathbb{R}^d, \mathbb{R})$ comme suit : $\mathscr{L}_\omega(w) = L_{+,\omega} w_1 + iL_{-,\omega} w_2$.

Les opérateurs $L_{+,\omega}$ et $L_{-,\omega}$ sont aussi coercifs en restriction à des sous-espaces de $H^1(\mathbb{R}^d, \mathbb{R})$ convenables qui dépendent du signe de $\frac{d}{d\omega} \int_{\mathbb{R}^d} Q_\omega(x)^2\, dx$, ou encore de la valeur de p. Remarquons d'ailleurs qu'en dimension 1, l'opérateur $L_{-,\omega}$ s'identifie à l'opérateur linéaire de (gKdV) qui correspond à $c = \omega$.

Définissons l'énergie linéarisée autour de Q_ω : pour tout $w \in H^1(\mathbb{R}^d)$, on pose :

$$\begin{aligned} H(w) &:= \operatorname{Re} \int_{\mathbb{R}^d} \mathscr{L}_\omega w \overline{w}\, dx, \\ &= \int_{\mathbb{R}^d} \left(|\nabla w|^2 + \omega |w|^2 - \left(Q_\omega^{p-1} |w|^2 + (p-1) Q_\omega^{p-1} w_1^2 \right) \right)\, dx. \end{aligned}$$

Les propriétés de coercivité des opérateurs linéarisés sont centrales dans la théorie non-linéaire de (gKdV) et de (NLS) autour des solitons, et particulièrement dans l'étude des propriétés qualitatives des solitons et des multi-solitons. Une des utilisations importantes de la coercivité concerne la stabilité orbitale des solitons et des multi-solitons ; on pourra consulter par exemple [3, Definition 4.1], respectivement [109], pour une définition de cette notion pour (gKdV), respectivement pour (NLS).

Rappelons que, dans le cas L^2-sous-critique, la coercivité de L_c est un élément clé de la preuve de la stabilité orbitale des solitons de (gKdV) [3,37,109]. La considération des opérateurs linéarisés montre aussi que les solitons sont instables dans les cas critique et surcritique [3, 37].

Dans le cas des équations de Schrödinger avec non-linéarité de type puissance, les travaux successifs et complémentaires concernant l'étude des opérateurs linéarisés et de la stabilité des solitons dus à Weinstein [108, 109], Grillakis [36], Grillakis, Shatah et Strauss [37,38], Maris [62], Schlag [100], Duyckaerts et Merle [24], Duyckaerts et Roudenko [26] et Côte, Martel et Merle [19] permettent d'obtenir :

Proposition 1.5. *Soit $\omega > 0$. Si $1 < p < 1 + \frac{4}{d}$, alors il existe $\mu_+ > 0$ tel que pour tout $w = w_1 + iw_2 \in H^1(\mathbb{R}, \mathbb{C})$,*

$$H(w) \geq \mu_+ \|w\|_{H^1}^2 - \frac{1}{\mu_+}\left(\left(\int_{\mathbb{R}^d} w_1 Q_\omega \, dx\right)^2 + \sum_{i=1}^{d}\left(\int_{\mathbb{R}^d} w_1 \partial_{x_i} Q_\omega \, dx\right)^2 + \left(\int_{\mathbb{R}^d} w_2 Q_\omega \, dx\right)^2\right) \tag{1.5}$$

et de plus Q_ω est orbitalement stable.
Si $1 + \frac{4}{d} < p < \frac{d+2}{d-2}$, alors $i\mathscr{L}\omega$ possède un vecteur propre Y_ω associé à une valeur propre $e_0 > 0$, il existe $\mu_+ > 0$ tel que pour tout $w = w_1 + iw_2 \in H^1(\mathbb{R}, \mathbb{C})$,

$$\begin{aligned} H(w) \geq \mu_+ \|w\|_{H^1}^2 &- \frac{1}{\mu_+}\left(\int_{\mathbb{R}^d} w_1 Y_2 \, dx\right)^2 \\ &- \frac{1}{\mu_+}\left(\sum_{i=1}^{d}\left(\int_{\mathbb{R}^d} w_1 \partial_{x_i} Q_\omega \, dx\right)^2 + \left(\int_{\mathbb{R}^d} w_2 Y_1 \, dx\right)^2 + \left(\int_{\mathbb{R}^d} w_2 Q_\omega \, dx\right)^2\right) \end{aligned} \tag{1.6}$$

et de plus Q_ω est instable.

Dans le cadre de l'étude des équations de Schrödinger avec non-linéarités plus générales, mentionnons qu'il est possible d'adapter les résultats de coercivité et de stabilité précédents (on pourra se reporter aux propositions 2.15 et 2.16 au Chapitre 2).

Pour l'étude plus spécifique des multi-solitons dans les cas non-intégrables où la méthode de scattering inverse ne s'applique pas, on recourt à des versions dépendantes du temps et localisées en espace de la coercivité des opérateurs linéarisés précédents, en se souvenant, par exemple pour (gKdV), que le multi-soliton se comporte essentiellement comme le $i^{\text{ème}}$ soliton au voisinage de $c_i t + x_i$ pour tout $i = 1, \ldots, N$.

Concrètement, afin de concentrer l'étude autour du point $c_i t + x_i$ en espace et ainsi d'observer le comportement de la solution u au voisinage du $i^{\text{ème}}$ soliton, cela suggère d'introduire des fonctions « cut off » ϕ_i.

Une telle famille de fonctions est typiquement générée de la manière suivante (voir aussi par exemple [80, paragraphe 2.2] ou [63, paragraphe 3.1]). On considère une fonction $\psi : \mathbb{R} \to \mathbb{R}$ de classe $\mathscr{C}^\infty$, strictement croissante et telle que

$$\lim_{-\infty} \psi = 0, \quad \lim_{+\infty} \psi = 1, \quad \forall\, x \in \mathbb{R}, \ \psi(x) = 1 - \psi(-x),$$

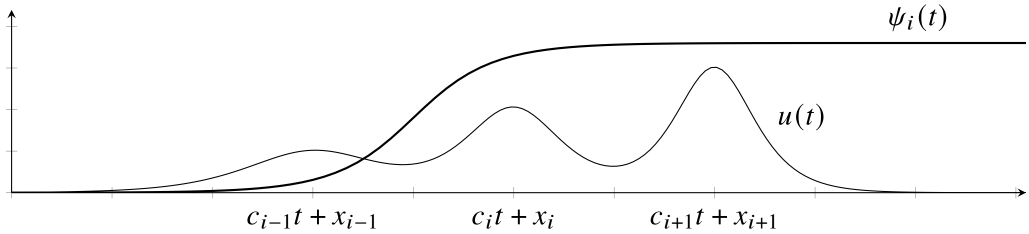
et

$$\forall\, x \in \mathbb{R}, \quad |\psi^{(3)}(x)| \leq \sigma_0 \psi'(x),$$

avec $\sigma_0 > 0$ suffisamment petit, dépendant des paramètres c_i. On précise qu'une telle fonction ψ existe effectivement. Pour tout $i \geq 2$, on définit alors

$$\psi_i(t,x) = \psi\left(x - \frac{c_{i-1}+c_i}{2}t - \frac{x_{i-1}+x_i}{2}\right).$$

Illustration : « filtration » des ondes solitaires par la fonction $\psi_i(t)$



En posant

$$\phi_1(t,x) = 1-\psi_2(t,x), \quad \phi_N(t,x) = \psi_N(t,x), \quad \phi_i(t,x) = \psi_i(t,x)-\psi_{i+1}(t,x), \quad i = 2,\dots,N-1$$

on se ramène à l'étude de quantités comme « l'énergie localisée au voisinage du $i^{\text{ème}}$ soliton », soit pour (gKdV) l'intégrale

$$\int_{\mathbb{R}} \left((\partial_x u)^2 + cu^2 - pR_i^{p-1}u^2\right)(t,x)\phi_i(t,x)\ dx.$$

Par exemple, $\int_{\mathbb{R}} u^2(t,x)\phi_i(t,x)\ \ dx$ correspond à la masse de la solution en restriction à un certain voisinage de $c_i t + x_i$ (à une précision $\mathrm{O}(e^{-\gamma t})$) et on peut établir un lien entre cette quantité et la masse du $i^{\text{ème}}$ soliton.

La structure particulière de (gKdV) permet d'obtenir la presque-monotonie de certaines quantités, définies à partir de u et de ψ_i ; en particulier,

$$\frac{d}{dt}\int_{\mathbb{R}} u^2(t,x)\psi_i(t,x)\ dx \leq Ce^{-\gamma t}.$$

Notons que les termes d'interaction entre les différents solitons sont essentiellement contrôlés par des inégalités de la forme

$$\forall\, i \neq j, \qquad \int_{\mathbb{R}} R_i R_j(t)\ dx \leq Ce^{-\gamma t},$$

où $\gamma > 0$ dépend des vitesses (et des fréquences pour (NLS)) des solitons R_i et R_j ; il est ici important de supposer que les vitesses sont deux à deux distinctes.

A l'aide des propriétés de coercivité et de monotonie comme ce qui précède, les résultats de stabilité de Martel, Merle et Tsai [80] des multi-solitons de (gKdV) ont ainsi pu voir le jour. En procédant de façon analogue, mais en prenant des fonctions ψ_i décroissantes, l'existence et l'unicité des multi-solitons de (gKdV) ont été démontrées dans les cas L^2-sous-critique et L^2-critique [63].

1.1.4 Existence et construction de multi-solitons pour (NLS) et (gKdV)

Première construction dans un cas non-intégrable

Dans son article [85] publié en 1990, Merle construit pour la première fois un multi-soliton dans le cadre d'une équation aux dérivées partielles non-intégrable, celle de Schrödinger non-linéaire L^2-critique

$$\partial_t u = i\left(\Delta u + |u|^{\frac{4}{d}} u\right). \tag{cNLS}$$

A titre d'illustration, notons qu'en dimension $d = 2$, l'équation précédente consiste en un modèle d'approximation satisfaisant pour décrire la propagation d'un faisceau laser planaire le long d'une seule direction t dans $\mathbb{R}^3$.

L'équation (cNLS) présente la caractéristique d'admettre la loi d'invariance pseudo-conforme suivante : si u est une solution de (cNLS), alors $(t,x) \mapsto \frac{1}{|t|^{\frac{d}{2}}} e^{i\frac{|x|^2}{4t}} \overline{u}\left(\frac{1}{t}, \frac{x}{t}\right)$ est également solution de (cNLS).

On observe en outre que les solutions de (cNLS) de la forme

$$q_{\omega,x_0}(t,x) = \frac{1}{(T-t)^{\frac{d}{2}}} e^{\frac{i}{T-t}\left(\frac{|x|^2}{4} - \omega\right)} Q_\omega\left(\frac{x-x_0}{T-t}\right), \tag{1.7}$$

où $\omega > 0$ et Q_ω vérifie $-\Delta Q_\omega + \omega Q_\omega - Q_\omega^{\frac{4}{d}+1} = 0$ explosent en temps fini T au point x_0. En fait, on peut montrer réciproquement que toutes les solutions de (cNLS) de masse minimale $\|Q_\omega\|_{L^2}$ qui explosent en temps fini T sont de la forme précédente [86, 87].

Merle [85] obtient l'existence de multi-solitons associés à K solitons comme corollaire d'un résultat d'existence de solutions u explosant en temps fini en K points $x_1, \dots, x_K$, construites à partir de solutions q_{ω_i,x_i} définies comme ce qui précède (1.7), en utilisant la loi d'invariance pseudo-conforme de (cNLS).

Une méthode de construction typique de multi-solitons à valeurs dans H^1

L'existence des multi-solitons pour les équations (gKdV) est garantie pour toute valeur de $p > 1$.

Théorème 1.6 (Merle [85] ; Martel [63] ; Côte, Martel et Merle [19]). *Soit $p > 1$, soit $N \in \mathbb{N} \setminus \{0,1\}$ et soient N solitons $R_{c_1,x_1}, \dots, R_{c_N,x_N}$ de (gKdV) associés à des paramètres de vitesse $c_1, \dots, c_N > 0$ deux à deux distincts.*
Il existe $T \in \mathbb{R}$ et $u \in \mathscr{C}([T,+\infty), H^1(\mathbb{R}))$ une solution de (gKdV) telle que

$$\left\| u(t) - \sum_{i=1}^{N} R_{c_i,x_i}(t) \right\|_{H^1} \to 0, \quad \textit{lorsque } t \to +\infty.$$

Se fondant sur les idées de Martel [63] qui démontre l'existence de multi-solitons pour (gKdV) dans les cas L^2-sous-critique et L^2-critique, le principe de la construction de multi-solitons à valeurs dans H^1, qui a été étendue par Côte, Martel et Merle [19] au cadre L^2-surcritique, est aujourd'hui bien connu.

La preuve de l'existence de multi-solitons dans H^1 a ainsi été établie pour un certain nombre d'équations aux dérivées partielles dispersives non-linéaires, dans chacun des cas stable et instable : pour (NLS) [19, 71, 85], pour les équations de Klein-Gordon non-linéaires [20], pour l'équation de Hartree [52], pour le système des « water-waves » [88], et récemment pour les équations de Zakharov-Kuznetsov [107].

Rappelons les étapes essentielles de la construction dans le cas où tous les solitons R_k considérés sont stables (c'est-à-dire lorsque $1 < p < 5$). Considérant une suite strictement croissante de réels (S_n) qui tend vers $+\infty$ et (u_n) une suite de solutions telles que $u_n = R(S_n)$, où

$$R = \sum_{k=1}^{K} R_k,$$

on démontre l'existence de $T_0 \in \mathbb{R}$ (indépendant de n) tel que pour tout n suffisamment grand,

$$\forall\, t \in [T_0, S_n], \qquad \|u_n(t) - R(t)\|_{H^1} \leq Ce^{-\gamma t}. \tag{1.8}$$

La majoration précédente, uniforme par rapport à n, est le point central de la construction. Elle repose sur un argument de type « bootstrap » qui fonctionne grâce à l'utilisation d'une propriété de coercivité satisfaite par une certaine fonctionnelle sous réserve que des conditions d'orthogonalité soient vérifiées. Cette fonctionnelle est construite à partir des opérateurs linéarisés autour des ground states et dépend du temps t ; aussi la propriété de coercivité utilisée est une version analogue aux propriétés exposées dans les propositions 1.4 et 1.5 mais dépendante de t. Notons qu'un argument de modulation des paramètres de la solution (s'appuyant sur le théorème des fonctions implicites) est nécessaire pour obtenir une nouvelle variable $z := u - \tilde{R}$ qui vérifie les conditions d'orthogonalité requises.
Ensuite, on montre que la suite $(u_n(T_0))_n$ d'éléments de H^1 vérifie une certaine propriété de compacité. On déduit de cette dernière et de (1.8) l'existence d'une fonction $\varphi_0 \in H^1$ telle que $(u_n(T_0))_n$ converge vers φ_0 (à extraction près) au sens de la norme L^2.
Enfin, la solution φ de l'équation considérée avec condition initiale φ_0 en T_0 est un multi-soliton qui vérifie la propriété de décroissance exponentielle. Cela peut se justifier notamment par des arguments de dépendance continue par rapport à la donnée initiale des solutions à valeurs dans H^s, $s \in [0, 1]$, sur des intervalles de temps compacts et par des arguments plus élémentaires de passage à la limite. Par exemple, Martel procède de la sorte dans [63] pour (gKdV), en utilisant les résultats de dépendance continue des solutions H^1 sur les compacts de Kenig, Ponce et Vega [50]. Martel et Merle [71] utilisent les résultats de Tsutsumi [105] de dépendance continue (par rapport à la donnée initiale) pour les solutions globales à valeurs dans L^2 en ce qui concerne l'équation (NLS) associée à la non-linéarité de type puissance avec $1 < p < 1 + \frac{4}{d}$ et aussi ceux de Cazenave et Weissler [7] pour les non-linéarités plus générales.

Dans le cas où l'un des solitons au moins n'est pas stable, le contrôle des directions instables nécessite la mise en place d'un argument supplémentaire, comprendre un argument topologique qui repose sur l'utilisation du théorème de Brouwer. On pourra se référer aux travaux de Côte, Martel et Merle [19], novateurs en cela.

Dans le même état d'esprit, la construction de trains infinis de solitons a été réalisée pour (NLS) par Le Coz et Tsai [60] et Le Coz, Li et Tsai [59] dans le cas H^1-sous-critique, moyennant certaines hypothèses de vitesses relatives élevées et de grandes fréquences pour les solitons. L'approche est en revanche différente de celle initiée par Martel puisque ces articles exploitent la formule de Duhamel, les inégalités de Strichartz et travaillent bien davantage avec les normes L^q.

1.1.5 Une autre interaction possible entre les ondes solitaires

Dans le cadre de la définition 1.3, les interactions entre solitons sont exponentiellement petites ; on parle d'interaction « faible ». Récemment, on a commencé à étudier des solutions pour lesquelles les interactions entre solitons sont algébriquement petites ; dans ce contexte, on dit que l'interaction est « forte ». Nous détaillons ici quelques résultats à ce sujet.

Pour (cNLS) en dimension 2, Martel et Raphaël [83, Theorem 1] démontrent l'existence de solutions globales qui se comportent asymptotiquement lorsque $t \to +\infty$ comme la somme de K ondes solitaires centrées en les sommets d'un K-gone régulier. Ces solutions ont une dynamique en temps long différente des multi-solitons de la définition 1.3 du paragraphe 1.1.2 ; elles ont la particularité notable d'exploser en temps infini : précisément, la norme L^2 du gradient de la solution évolue ici en $\log t$. De surcroît, les K ondes solitaires ne sont pas découplées en temps long comme dans le cas d'une interaction faible prévue par la définition 1.3. Dit autrement, les solutions obtenues par invariance pseudo-conforme de (cNLS) à partir des solutions construites concentrent les K « bulles » en un même point au temps limite d'explosion.

Un autre exemple d'interaction forte apparaît en dimension 1 dans le contexte des équations (gKdV). Nguyen [94] construit pour $2 < p < 5$ et $p > 5$ un 2-soliton en $+\infty$ pour lequel les deux ondes solitaires associées se propagent à la même vitesse et dont la distance relative est en $2 \log t + \text{cte}$. Pour une certaine constante c_p que l'on peut expliciter en fonction de la puissance p de la non-linéarité, il existe une solution u telle que

$$\left\| u(t) - Q\left(\cdot - t - \log(c_p t)\right) - \sigma Q\left(\cdot - t + \log(c_p t)\right)\right\|_{H^1} \to 0 \quad \text{lorsque } t \to +\infty,$$

où $\sigma = -1$ en régime sous-critique et $\sigma = 1$ dans le cas surcritique.

Prolongeant les résultats précédents et les méthodes mises à l'œuvre dans les preuves correspondantes, l'article [95] révèle l'existence de solutions de l'équation de Schrödinger (NLS) qui se comportent comme une somme de deux solitons séparés d'une distance qui évolue également en $2 \log t$ lorsque $t \to +\infty$, dans les cas L^2-sous-critique et L^2-surcritique. En outre, pour le système d'équations de Schrödinger cubiques unidimensionnelles couplées, Martel et Nguyen [82] obtiennent des 2-solitons symétriques avec distance relative logarithmique (précisément en $\log t + \frac{1}{2} \log \log t$). Dans le même papier, les deux auteurs exhibent également une nouvelle dynamique qui n'apparaît notamment pas dans le cas de l'équation intégrable (NLS) cubique : deux solitons de vitesses différentes (donc d'amplitudes différentes) peuvent interagir de manière forte en $\mathrm{O}(\log t)$.

Des solutions composées de deux ou plusieurs solitons interagissant de manière forte ont aussi fait récemment l'objet d'étude pour l'équation des ondes critique par Jendrej [44] et Jendrej et Martel [46].

Les dynamiques logarithmiques précédentes semblent très particulières et plutôt exceptionnelles. En ce qui concerne les questions relatives à la classification, mentionnons que l'article [43] de Jendrej signale pour (gKdV) L^2-surcritique que des 2-solitons associés à deux ondes solitaires de même amplitude ne peuvent exister que si ces dernières sont séparées d'une distance adéquate en $\mathrm{O}(\log t)$. De même, pour l'équation des wave maps (à valeurs dans des variétés plus générales que $\mathbb{R}$ ou $\mathbb{C}$), on pourra consulter l'article [45] qui fournit un résultat d'unicité dans une certaine classe pour les solutions particulières issues de l'interaction forte entre les ondes solitaires.

Ces solutions obtenues récemment font l'objet de constructions souvent assez délicates. Des résultats d'existence analogues, inspirés des travaux ci-dessus, sont à ce jour au stade de conjecture pour les équations de Zakharov-Kuznetsov [106]. Plus généralement, beaucoup de propriétés concernant les solutions de la nature décrite sont à explorer dans les différentes situations relatées. Toutefois, nous ne nous focaliserons pas davantage sur ces solutions « exceptionnelles » dont l'existence repose sur l'interaction forte entre les solitons. Aussi, dans cette thèse, nous limiterons notre étude au cadre de la définition 1.3.

1.2 Propriétés des multi-solitons des équations de Schrödinger non-linéaires

1.2.1 Etude de la régularité des solutions

Afin d'étudier la régularité H^s des multi-solitons de (NLS), nous nous appuyons considérablement sur les résultats d'existence et d'unicité de solutions générales de l'équation dans H^s, pour tout $s \geq 1$. L'étude préalable du problème de Cauchy pour (NLS) dans H^s s'avère donc indispensable.

Problème de Cauchy pour (NLS) dans H^s

Le problème de Cauchy pour (NLS) a été étudié à l'aide d'arguments de points fixes appliqués à différents espaces bien choisis. Après les résultats qui ont été obtenus dans $H^1(\mathbb{R}^d)$ [6, 34] et, plus spécifiquement pour $1 < p < 1 + \frac{4}{d}$ dans $L^2(\mathbb{R}^d)$ [91, 105], Cazenave et Weissler [7] ont montré que, sous des hypothèses H^s-sous-critiques convenables, (NLS) est localement bien posé dans H^s ($s > 1$) avec dépendance continue dans H^s par rapport à la donnée initiale sur certains compacts.

Théorème 1.7. *Si $s \geq \frac{d}{2}$, ou si $0 \leq s < \frac{d}{2}$ et $1 < p < 1 + \frac{4}{d-2s}$, alors pour tout $u_0 \in H^s(\mathbb{R}^d)$, il existe $T_{max}(u_0) > 0$ et une unique solution $u \in \mathscr{C}((-T_{max}(u_0), T_{max}(u_0)), H^s(\mathbb{R}^d))$ de* (NLS) *telle que $u(0) = u_0$.*
De plus, il existe $T \in (0, T_{max}(u_0))$ tel que pour toute suite $(u_0^n)_n$ qui converge vers u_0 dans $H^s(\mathbb{R}^d)$, la solution u^n de (NLS) *telle que $u^n(0) = u_0^n$ est définie sur [-T,T] pour n large et (u^n) converge vers u dans $\mathscr{C}([-T, T], H^s(\mathbb{R}^d))$.*

L'énoncé démontré par Cazenave et Weissler est en réalité plus précis en ce qui concerne le résultat d'existence et d'unicité locales, dans la mesure où les auteurs considèrent également des espaces auxiliaires (de Besov). Notons qu'en passant cette fois par des espaces de Lebesgue, Kato [48] a généralisé le théorème précédent aux non-linéarités de classe $\mathscr{C}^s$ (au sens des fonctions différentiables $\mathbb{R}^2 \to \mathbb{R}^2$) dont les dérivées partielles d'ordre $s' \in \{0, \ldots, s\}$ croissent en $\mathrm{O}\left(r^{k-s'}\right)$ pour un certain $k \in \left[s, 1 + \frac{4}{d-2s}\right]$ dans le cas où $s \leq \frac{d}{2}$.

Notons que le résultat de dépendance continue par rapport à la donnée initiale a été affiné par Dai, Yang et Cao [21]. En s'appuyant sur les énoncés de Cazenave et Weissler et de Kato, on peut démontrer que la dépendance continue dans $H^s(\mathbb{R}^d)$ est valide sur tout compact [21].

En reprenant les notations du théorème 1.7, on a :

Théorème 1.8. *Pour tout segment* $[-S, S]$ *inclus dans* $(-T_{max}(u_0), T_{max}(u_0))$*, pour toute suite* $(u_0^n)_n$ *qui converge vers* u_0 *dans* $H^s(\mathbb{R}^d)$*, la solution* u^n *de* (NLS) *telle que* $u^n(0) = u_0^n$ *est définie sur* $[-S, S]$ *et converge vers* u *dans* $\mathscr{C}([-S, S], H^s(\mathbb{R}^d))$.

Le théorème précédent présente un intérêt majeur : comme le problème de Cauchy est localement bien posé dans $H^s(\mathbb{R}^d)$ avec dépendance continue sur tout compact, beaucoup de calculs que nous réalisons (comme les dérivées d'intégrales à paramètres de fonctions H^s) et qui sont présentés formellement peuvent être justifiés par des arguments de régularisation.

Multi-solitons H^s

Nous étudions ici la question naturelle de la régularité des multi-solitons. Peut-on espérer en particulier l'appartenance des multi-solitons construits à valeurs dans H^1 à des espaces de Sobolev supérieurs ?
Les travaux de Martel [63] apportent une réponse affirmative dans le cas de (gKdV), unidimensionnelle. Il est démontré dans [63] que la convergence exponentielle (1.8) a lieu au sens de la norme H^s, pour tout $s \in \mathbb{N}$, avec une constante C_s devant l'exponentielle qui dépend de s et un taux de décroissance exponentielle θ indépendant de s. Notons que des estimées analogues ont été obtenues très récemment pour les équations de Zakharov-Kuznetsov par Valet dans [107] en faisant l'usage d'un argument qui utilise la monotonie typique des équations (gKdV).

Un résultat important de cette thèse est la construction pour (NLS) avec non-linéarité générale

$$\partial_t u = i\left(\Delta u + f(|u|^2)u\right) \tag{NLS}$$

d'un multi-soliton à valeurs dans $H^s(\mathbb{R}^d)$ où l'indice de régularité $s > 1$ est intimement lié à la régularité de la fonction $g : z \mapsto zf(|z|^2)$.

Plus exactement, dans le cas particulier de la non-linéarité de type puissance, le théorème démontré au chapitre 2 prend la forme suivante.

Théorème 1.9 (Côte, F.). *Supposons* $p \geq 3$. *Soit* $s_0 = \lfloor p-1 \rfloor \geq 2$ *ou* $s_0 = +\infty$ *si* p *est un entier impair. Il existe* $\theta > 0$, $T_1 > 0$ *et* $u \in \mathscr{C}([T_1, +\infty), H^{s_0}(\mathbb{R}^d))$ *solutions de* (NLS) *avec non-linéarité de type puissance* $f : r \mapsto r^{\frac{p-1}{2}}$ *tels que pour tout entier positif* $s \leq s_0$*, il existe une constante* $C_s \geq 1$ *telle que*

$$\forall\, t \geq T_1, \qquad \|u(t) - R(t)\|_{H^s} \leq C_s e^{-\frac{2\theta}{s+1}t}.$$

Par ailleurs, si p *est un entier impair, alors pour tout* $s \geq 0$,

$$\forall\, t \geq T_1, \qquad \|u(t) - R(t)\|_{H^s} \leq \sqrt{C_s} e^{-\theta t}.$$

Précisons que la solution u précédente est *a priori* différente de celle du théorème 1.6, obtenue dans H^1.

Le théorème 1.9 qui concerne la non-linéarité particulière de type puissance se limite en réalité aux dimensions inférieures ou égales à 3 en raison de l'hypothèse H^1-sous-critique $p < 1 + \frac{4}{(d-2)_+}$ faite pour assurer l'existence de solitons.

Nous disposons d'un résultat de régularité analogue au théorème 1.9 pour des non-linéarités générales. Sous l'hypothèse que g est un élément de $W^{s_0+1,\infty}_{loc}(\mathbb{C})$, que l'on a $s_0 > \frac{d}{2}$ et que chaque opérateur linéarisé autour des solitons Q_{ω_k} vérifie une propriété de coercivité adéquate qui traduit en quelque sorte la stabilité ou l'instabilité du soliton considéré (voir Proposition 1.5), nous démontrons que la conclusion du théorème 1.9 est encore vraie dans ce cadre plus général.

La démonstration des résultats de régularité précédents, que l'on trouvera au chapitre 2, est écrite pour des non-linéarités g quelconques. Soulignons qu'elle ne consiste pas en un travail direct sur le multi-soliton H^1 qui a été construit dans [71] ou [19]. Elle repose en revanche sur la preuve d'estimées H^s-uniformes satisfaites par une suite (u_n) de solutions bien choisie, qui était déjà considérée dans [71] et [19] et sur la mise en place d'un argument usuel de compacité. La conclusion de la preuve se fait à l'aide de la dépendance continue sur tout compact des solutions de (NLS) par rapport aux données initiales décrite par le théorème 1.8. Prenant comme point de départ les estimées H^1-uniformes obtenues par Martel et Merle et par Côte, Martel et Merle, et en s'inspirant de Martel [63, section 3], la preuve des estimées H^s se fait par récurrence sur l'indice de régularité de la fonction g.
Le point clé de la récurrence est d'observer que la fonctionnelle suivante

$$G_{n,s} : t \mapsto \int_{\mathbb{R}^d} \left\{ \sum_{|\alpha|=s} \binom{s}{\alpha} |\partial^\alpha u_n|^2 - \sum_{|\beta|=s-1} \binom{s-1}{\beta} \mathrm{Re}\left(u_n^2 \left(\partial^\beta \overline{u_n}\right)^2\right) f'(|u_n|^2) \right\}(t)\ dx,$$

qui s'écrit de la même manière en toute dimension d, peut être contrôlée par $\|u_n(t) - R(t)\|_{H^{s-1}}$. L'introduction de cette fonctionnelle permet d'éliminer, lorsqu'on la dérive, les termes que l'on ne sait pas contrôler dans la dérivée de $\|u_n(t)\|_{H^s}^2$, à savoir

$$\int_{\mathbb{R}^d} \mathrm{Im}\left(u_n^2 (\partial^\alpha \overline{u_n})^2\right) f'(|u_n|^2)\ dx$$

pour les multi-indices $\alpha \in \mathbb{N}^d$ de longueur s. De la sorte, la dérivée première de $G_{n,s}$ ne contient ni terme quadratique faisant intervenir $\partial^\alpha u_n$ avec $|\alpha| = s$ ni terme $\partial^{\alpha'} u_n$ avec $|\alpha'| > s$, ce qui est essentiel pour pouvoir appliquer l'hypothèse de récurrence.
En pratique, le contrôle des différents termes qui apparaissent dans $G'_{n,s}$ est réalisé à l'aide d'outils classiques d'analyse fonctionnelle, notamment l'inégalité de Hölder, les injections de Sobolev et les inégalités d'interpolation de Gagliardo-Nirenberg.

Notons que si pour (gKdV), Martel [63] parvient à montrer que toutes les quantités $\|u_n(t) - R(t)\|_{H^s}$ décroissent exponentiellement en temps long avec la même vitesse, nous obtenons, dans le cadre de (NLS), des taux de décroissance exponentielle qui sont divisés par deux pour ces mêmes quantités lorsque l'on passe de s à $s+1$. En effet, il ne nous est guère possible de considérer autant d'intégrations par parties, dans la mesure où la non-linéarité n'est pas nécessairement indéfiniment dérivable et où surtout l'algèbre associée à (NLS) n'est pas aussi favorable. Plus précisément, pour (gKdV), les termes qui apparaissent du type $\partial^{s+1} u_n \partial^s u_n$ peuvent être intégrés tandis que pour (NLS), les termes analogues sont de la forme $\mathrm{Im}\left(\partial^{s+1} u_n\right) \mathrm{Re}\left(\partial^s u_n\right)$ (en dimension 1) et ne peuvent ainsi pas être traités de la même façon.

Ainsi, nous notons la dépendance en s dans les estimées du théorème 1.9 liée à la perte en ce qui concerne la vitesse de décroissance exponentielle lorsqu'il s'agit de passer de la preuve de l'estimée H^s à celle de l'estimée H^{s+1}. Le taux de décroissance exponentielle de la norme H^s de $u(t) - R(t)$, correspondant à $\frac{2\theta}{s+1}$ et obtenu finalement par interpolation, est optimal pour $s \geq 2$ (qui est le cas intéressant du théorème précédent).

Rajoutons que notre preuve est aussi plus technique en raison de la dimension. Comme les éléments de $H^1(\mathbb{R}^d)$ ne sont pas nécessairement dans $L^\infty(\mathbb{R}^d)$ pour $d \geq 2$, nous recourons à un argument de type « bootstrap » afin de démontrer les estimées H^s-uniformes souhaitées. L'hypothèse $s_0 > \frac{d}{2}$ est déjà utile à ce niveau.

Par ailleurs, se pose la question de l'hypothèse optimale à fournir quant à la non-linéarité g de sorte à obtenir un multi-soliton u dans $H^{s_0}(\mathbb{R}^d)$ tel que $\|u(t) - R(t)\|_{H^{s_0}}$ soit à décroissance exponentielle.

En l'occurrence, justifions qu'il convient de supposer que g soit dans l'espace $W^{s_0+1,\infty}_{loc}(\mathbb{C})$. Rappelons qu'il suffisait de prendre g de classe $\mathscr{C}^1$ (considérée comme fonction $\mathbb{R}^2 \to \mathbb{R}^2$) pour avoir le résultat avec $s_0 = 1$. D'autre part, dans notre cadre d'étude de (NLS) où l'on exige $s_0 > \frac{d}{2}$, le problème de Cauchy est localement bien posé dans H^{s_0} sous l'hypothèse que g est de classe $\mathscr{C}^{s_0}$ (Kato, [48]). Au minimum, il est donc souhaitable de faire cette dernière hypothèse.
Cela dit, afin d'obtenir les inégalités voulues, il est incontournable de savoir contrôler les quantités de la forme

$$\frac{\partial^{s_0} g}{\partial_x^r \partial_y^{s_0-r}}(u) - \frac{\partial^{s_0} g}{\partial_x^r \partial_y^{s_0-r}}(R),$$

où $r = 0, \ldots, s_0$. Il semble donc naturel de supposer que $\frac{\partial^{s_0} g}{\partial_x^r \partial_y^{s_0-r}}$ soit localement lipschitzienne sur $\mathbb{C}$ (ou, autrement dit, appartienne à $W^{1,\infty}_{loc}(\mathbb{C})$). D'autre part, le contrôle de l'intégrale qui fait intervenir typiquement les dérivées d'ordre maximal s_0 des deux quantités $u_n - R$ et g nécessite une intégration par parties de sorte que la quantité $u_n - R$ apparaisse avec une dérivée d'ordre $s_0 - 1$, ceci afin de pouvoir appliquer l'hypothèse de récurrence. Pour un multi-indice $\alpha = (\alpha_1, \ldots, \alpha_d)$ donné de longueur s_0 et de composante $\alpha_l \neq 0$, pour une fonction h donnée de classe $\mathscr{C}^1$ et pour $r = 0, \ldots, s_0$, nous pouvons écrire

$$\int_{\mathbb{R}^d} h(R_k) \frac{\partial^{s_0} g}{\partial_x^r \partial_y^{s_0-r}}(R_k) \partial^\alpha \overline{v} \, dx = -\int_{\mathbb{R}^d} \partial^{\alpha - e_l} \overline{v} \partial^{e_l} \left(h(R_k) \frac{\partial^{s_0} g}{\partial_x^r \partial_y^{s_0-r}}(R_k) \right) dx \tag{1.9}$$

(avec e_l le $l^{\text{ième}}$ vecteur de la base canonique de $\mathbb{R}^d$) sous réserve que la dérivée de $x_l \mapsto \frac{\partial^{s_0} g}{\partial_x^r \partial_y^{s_0-r}}(R_k)$ appartienne à un certain espace de Lebesgue convenable. Ce point est assuré dès lors que cette dérivée est bornée.

Notre démonstration du résultat de régularité s'appuie manifestement sur l'hypothèse $s_0 > \frac{d}{2}$ qui, remarquons-le, est automatique en dimension $d \leq 3$. Cette dernière, utilisée dans l'argument bootstrap afin de travailler avec des éléments de $L^\infty(\mathbb{R}^d)$ comme évoqué plus haut, apparaît aussi naturellement lorsqu'il y a lieu de contrôler les différents termes dans l'expression de $G'_{n,s}$.

Probablement qu'une autre méthode, voire d'autres outils, seraient à mettre en œuvre pour répondre à la question de la régularité dans le cas où $s_0 \leq \frac{d}{2}$. Pour l'heure, il semblerait que les inégalités de Strichartz usuelles associées à (NLS) ne permettent pas de répondre positivement au problème (sauf si les vitesses relatives sont grandes), mais nous pouvons raisonnablement penser que des estimées de ce type, qui concerneraient plus particulièrement l'équation linéarisée autour d'une somme de solitons, bien qu'inexistantes actuellement, pourraient s'avérer utiles dans ce cadre. D'ailleurs, dans une autre perspective, le fait d'avoir à disposition de telles estimées serait de bon présage pour obtenir ou réobtenir des résultats de stabilité ou de stabilité asymptotique en ce qui concerne les multi-solitons de (NLS).

1.2.2 Question de l'unicité des multi-solitons

Sous l'impulsion de [63] qui fournit l'existence et l'unicité des multi-solitons de (gKdV) dans les cas L^2-sous-critique et critique puis de [19] qui donne l'ingrédient nécessaire à la construction dans le cas L^2-surcritique, les travaux de Combet [11] permettent d'apporter une réponse complète à la question de la classification des multi-solitons de (gKdV). Au contraire, pour (NLS) et pour la plupart des autres équations aux dérivées partielles non-linéaires dispersives focalisantes, la classification exhaustive des multi-solitons n'est pas encore comprise à ce jour. Notons que les travaux de Valet [107] attachés à l'étude des multi-solitons de l'équation de Zakharov-Kuznetsov (ZK) fournissent aussi l'unicité dans les cas L^2-sous-critique et L^2-critique, mais des informations cruciales liées au spectre de l'opérateur linéarisé pour (ZK) manquent encore pour démontrer l'existence, voire espérer une classification des multi-solitons, en régime surcritique.
De même que pour (gKdV) [11], il est connu de Combet [12] l'existence d'une famille à K paramètres de solutions pour (NLS) dans le cas L^2-surcritique en dimension 1. En particulier, il n'y a pas unicité pour les multi-solitons associés à K solitons donnés dans ce cadre, mais pour autant, il n'est pas clair que l'on puisse obtenir tous les multi-solitons de la manière décrite dans [12], à la différence de (gKdV). Par ailleurs, un premier résultat d'unicité pour (NLS) a vu le jour dans les papiers [60] et [59]. Au même titre que pour la partie existence, ce résultat exige des vitesses relatives suffisamment élevées ; de plus, l'unicité démontrée par les trois auteurs Le Coz, Li et Tsai est valide dans une classe de solutions u telles que $u(t) - R(t)$ est à décroissance exponentielle en norme L^q, Strichartz ou H^1.

En nous inspirant de [63] et de [81], nous obtenons un autre résultat d'unicité pour (NLS), au sens de la norme H^1, qui concerne les cas stable et L^2-critique. C'est aussi un résultat d'unicité « avec condition », mais au sens où l'unicité a lieu dans la classe des multi-solitons u tels que $\|u(t) - R(t)\|_{H^1}$ décroît plus rapidement qu'une certaine puissance de $\frac{1}{t}$ pour t assez grand.

Exposons le résultat d'unicité obtenu lorsqu'on considère la non-linéarité particulière de type puissance.

Théorème 1.10 (Côte, F.). *Soit $d \leq 2$ et supposons que $p \in \left[3, 1 + \frac{4}{d}\right]$. Il existe $N \in \mathbb{N}$ et une* unique *solution $u \in \mathscr{C}([T_1, +\infty), H^1(\mathbb{R}^d))$ de (NLS) telle que*

$$\|u(t) - R(t)\|_{H^1} = \mathrm{O}\left(\frac{1}{t^N}\right), \qquad \text{lorsque } t \to +\infty. \tag{1.10}$$

Une conséquence directe, importante et rassurante, est que les multi-solitons des théorèmes 1.6 et 1.9 coïncident (et nous pouvons prendre en outre $T_0 = T_1$).

Là encore, le théoreme 1.10 possède en dimension $d \leq 3$ une généralisation aux non-linéarités f telles que, en notant $\tilde{f}(z) := f(|z|^2)$ pour tout $z \in \mathbb{C}$,

- la fonction $\tilde{f}$ est de classe $\mathscr{C}^2$ sur $\mathbb{C}$ vue comme fonction différentiable $\mathbb{R}^2 \to \mathbb{R}$.
- la différentielle seconde de $\tilde{f}$ est contrôlée de la manière suivante :

$$\left\|D_z^2 \tilde{f}\right\| = \mathrm{O}\left(|z|^{\frac{4}{d}-2}\right), \qquad \text{lorsque } |z| \to +\infty \ ;$$

- si $d \geq 2$, g appartient à $W^{s_0+1,\infty}_{loc}(\mathbb{C})$ où $s_0 := \lfloor \frac{d}{2} \rfloor + 1$;
- les opérateurs linéarisés autour des solitons Q_{ω_k} vérifient la propriété de coercivité (1.5) appropriée à la situation de stabilité.

Notons qu'il semble nécessaire de se restreindre aux dimensions $d \leq 3$; nous utilisons en effet que $\frac{4}{d} - 1 > 0$ pour assurer l'intégrabilité d'une certaine quantité et nous avons besoin de l'injection de Sobolev $H^1(\mathbb{R}^d) \hookrightarrow L^6(\mathbb{R}^d)$ pour contrôler les normes L^6 de quantités (non-bornées) par les normes H^1 correspondantes.

En dimension $d \geq 4$, nous pouvons aussi donner un énoncé similaire qui s'applique aux non-linéarités générales, mais alors au prix de considérer une classe de solutions u plus petite, pour laquelle un certain contrôle de $\int_t^{+\infty} \|u(s) - R(s)\|_{L^\infty}\, ds$ est réalisé.

Il s'avère que le résultat de régularité obtenu (Théorème 1.9) est utile dans notre preuve d'unicité en dimension $d \geq 2$. Pour $d \geq 2$, nous utilisons le multi-soliton construit à valeurs dans $H^{s_0}(\mathbb{R}^d)$.

L'idée de la preuve du théorème d'unicité est de montrer que la différence z entre une solution satisfaisant les hypothèses et le multi-soliton régulier construit dans le théorème 1.9 est nulle. Pour cela, nous considérons une certaine fonctionnelle d'énergie de type Weinstein en une nouvelle variable $\tilde{z}$, obtenue à partir de z par modulation, comme dans [63] (l'intérêt de la modulation est d'assurer des conditions d'orthogonalité requises pour la coercivité de H sur laquelle nous fondons notre preuve). Cette fonctionnelle tire son inspiration de [81] et s'écrit :

$$H(t) := \sum_{k=1}^{K} \int_{\mathbb{R}^d} \Big\{ |\nabla \tilde{z}|^2 - \left(f\left(|R_k|^2\right) |\tilde{z}|^2 + 2\mathrm{Re}(\overline{R_k}\tilde{z})^2 f'\left(|R_k|^2\right)\right) + \left(\omega_k + \frac{|v_k|^2}{4}\right) |\tilde{z}|^2 - v_k \cdot \mathrm{Im}\left(\nabla \tilde{z}\bar{\tilde{z}}\right) \Big\} \phi_k(t,x)\, dx. \quad (1.11)$$

Le travail sur la variable $\tilde{z}$ semble aussi mieux adapté au cadre particulier de (NLS) considéré ici.
Soulignons que, si un contrôle en $\mathrm{O}\left(e^{-\gamma t}\|z\|^2_{H^1}\right)$ permet d'obtenir l'unicité dans le cadre le plus général pour (gKdV), le contrôle de la dérivée de H par $\mathrm{O}\left(\frac{1}{t}\|\tilde{z}\|^2_{H^1}\right)$ explique que l'on doive se restreindre à la classe vérifiant (1.10). En raison du manque d'estimées exponentiellement décroissantes en temps (qui sont intéressantes dans la mesure où l'on ne perd rien lorsqu'on intègre en temps long), les différentes inégalités démontrées et utiles pour conclure dans ce cadre sont moins fortes que celles de [63] pour (gKdV) et demandent par conséquent une analyse plus fine. L'inégalité typique obtenue en rassemblant toutes les majorations intermédiaires et qui permet de conclure la preuve prend la forme

$$\|\tilde{z}(t)\|_{H^1} \leq \frac{C}{t} \int_t^{+\infty} \|\tilde{z}(s)\|_{H^1}\, ds.$$

Il ne semble pas clair de pouvoir améliorer cette inégalité avec la méthode que nous employons, fondée sur l'analyse de la dérivée de H.

Néanmoins, nous conjecturons que le résultat d'unicité précédent s'étend à toute la classe des solutions u telles que

$$\|u(t) - R(t)\|_{H^1} \to 0 \quad \text{lorsque} \quad t \to +\infty.$$

Pour l'heure, il est malgré tout remarquable d'observer que le théorème 1.10 permet de dépasser le cadre de la classe à décroissance exponentielle, laquelle demeure communément la classe naturelle de construction des multi-solitons.

1.3 Solutions non-dispersives des équations de Korteweg-de Vries généralisées

1.3.1 Solutions non-dispersives et théorèmes de rigidité pour (gKdV)

Les équations de Korteweg-de Vries généralisées présentent la caractéristique d'admettre des propriétés de type Liouville au voisinage des solitons. Un des premiers résultats en la matière remonte à [65, Theorem 1] dans le contexte de l'équation L^2-critique : l'approche repose sur des considérations et des techniques propres à (gKdV).

Puis, ce résultat a été étendu par Martel et Merle au cas L^2-sous-critique [66, 70] et, en faisant l'hypothèse d'une proximité suffisante d'un soliton en tout temps, au cas instable avec des non-linéarités générales [72].

Le théorème suivant regroupe les différentes situations.

Théorème 1.11 (Propriété de Liouville au voisinage d'un soliton; Martel et Merle [65, 66, 70, 72])**.** *Soit* $c_0 > 0$*. Il existe* $\alpha > 0$ *tel que si* $u \in \mathscr{C}(\mathbb{R}, H^1(\mathbb{R}))$ *est une solution de (gKdV) qui satisfait, pour une certaine fonction* $y : \mathbb{R} \to \mathbb{R}$ *de classe* $\mathscr{C}^1$*,*

$$\text{(proximité du soliton)} \qquad \forall\, t \in \mathbb{R}, \qquad \|u(t, \cdot + y(t)) - Q_{c_0}\|_{H^1} \leq \alpha, \tag{1.12}$$

$$\text{(non-dispersion)} \qquad \forall\, \varepsilon > 0,\ \exists\, R > 0,\ \forall\, t \in \mathbb{R}, \qquad \int_{|x|>R} u^2(t, x + y(t))\, dx \leq \varepsilon, \tag{1.13}$$

alors il existe $c_1 > 0$*,* $x_1 \in \mathbb{R}$ *tels que*

$$\forall\, t, x \in \mathbb{R}, \qquad u(t,x) = Q_{c_1}(x - x_1 - c_1 t).$$

Autrement dit, les solutions globales de (gKdV) qui sont suffisamment proches pour tout temps d'un état fondamental, mais qui ne sont pas des solitons, doivent nécessairement disperser au sens où la propriété de L^2-compacité (1.13) n'est pas satisfaite. Pour de telles solutions, la masse n'est plus concentrée en tout temps t autour de $y(t)$, et ceci aussi loin que l'on veut de $y(t)$. Le théorème précédent suggère que les solitons sont des objets très rigides.

À chaque fois, la stratégie mise en place pour prouver les théorèmes de Liouville précédents est la suivante : se ramener au cadre linéaire et y montrer une propriété de rigidité analogue qui concerne l'équation linéarisée autour d'un soliton, en utilisant toutefois des outils d'analyse non-linéaire (arguments du Viriel, de monotonie) qui consistent à étudier la dérivée de quantités du type $\int_{\mathbb{R}} v^2(t,x)\phi(x)\, dx$ pour des choix judicieux de fonctions v et ϕ. S'il est possible de travailler avec $\phi(x) = x$, les preuves, à l'instar de [64] et [72], ont progressivement gagné en efficacité en considérant $\phi(x) = -\frac{Q'_{c_0}(x)}{Q_{c_0}(x)}$ et une variable v, « duale » de u, construite à l'aide de l'opérateur linéarisé autour de Q_{c_0} et obtenue après modulation. Dans [72], Martel et Merle travaillent ainsi avec la variable

$$v := L_{c_0}\eta - \left(f(Q_{c_0} + \eta) - f(Q_{c_0}) - \eta f'(Q_{c_0}) \right),$$

où η a pour expression $\eta(t,x) := u(t,x+\rho(t)) - Q_{c(t)}(x)$ pour de certaines fonctions de modulation $\rho(t)$ et $c(t)$ de classe $\mathscr{C}^1$.

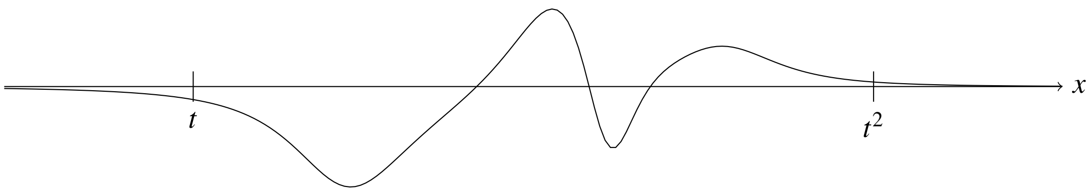
Notons que les propriétés de stabilité asymptotique H^1 des solitons et des multi-solitons ont été développées en parallèle des résultats de rigidité précédents pour la raison que ces derniers permettent de récupérer la stabilité asymptotique des solitons [66,70,72] et des multi-solitons [72,80]. À notre connaissance, très peu d'articles démontrent des résultats de stabilité asymptotique pour (gKdV) sans l'appui d'une propriété de type Liouville (l'article [97] est en cela un des rares exemples).

Notre propos est de fournir une propriété de Liouville analogue au théorème 1.11, adaptée aux multi-solitons. Nous considérons des solutions de (gKdV) qui sont non-dispersives au sens de la définition suivante.

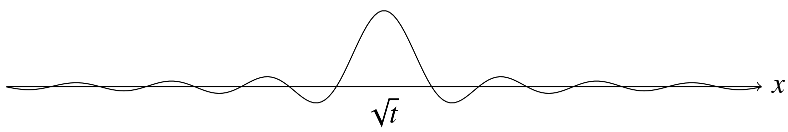
Définition 1.12. *On dit qu'un élément $u \in \mathscr{C}([T,+\infty), L^2(\mathbb{R}))$ est non-dispersif s'il existe $\rho > 0$ tel que*

$$\forall\, \varepsilon > 0,\ \exists\, R_\varepsilon > 0,\ \forall\, t \geq T, \qquad \int_{x<\rho t - R_\varepsilon} u^2(t,x)\, dx \leq \varepsilon. \tag{1.14}$$

Intuitivement, une solution de (gKdV) non-dispersive au sens de la définition précédente est une solution qui ne possède pas de masse à gauche en espace et pour laquelle toute la masse se déplace à droite en espace lorsque t augmente : observons en effet que la masse est essentiellement localisée à droite de ρt.



Exemple de fonction « non-dispersive » pour $t > 0$ au sens de (1.14)



Exemple de fonction qui « disperse » pour $t > 0$ au sens de (1.14) : $(t,x) \mapsto \frac{\sin(x-\sqrt{t})}{x-\sqrt{t}}$

A titre d'exemple, une fonction $u \in \mathscr{C}([T,+\infty), L^2(\mathbb{R}))$ satisfaisant

$$\forall\, t \geq T,\ \forall\, x \leq \rho t, \quad |u(t,x)| \leq C e^{-\kappa|x-\rho t|} \tag{1.15}$$

pour certaines constantes C et κ, est non-dispersive.
Nous démontrons réciproquement que toute solution $u \in \mathscr{C}([T,+\infty), H^1(\mathbb{R}))$ de (gKdV) non-dispersive et uniformément bornée dans $H^1(\mathbb{R})$ vérifie (1.15). Plus encore, il s'agit d'un élément

de $\mathscr{C}^\infty([T,+\infty)\times\mathbb{R})$ pour lequel toutes les dérivées partielles vérifient une inégalité de la forme (1.15).
C'est une observation fondamentale, dont la preuve s'appuie sur une inégalité de monotonie forte, inspirée de Martel et Merle [66] et de Laurent et Martel [57].

Nous prouvons au chapitre 3 que de telles solutions qui sont uniformément proches d'une somme de N solitons découplés sont des multi-solitons.

Théorème 1.13 (F.)**.** *Soit u une solution de (gKdV) qui appartient à $\mathscr{C}([0,+\infty),H^1(\mathbb{R}))$. Supposons l'existence de $\rho>0$ tel que*

$$\forall\,\varepsilon>0,\ \exists\,R_\varepsilon>0,\ \forall\,t\geq 0,\qquad \int_{x<\rho t-R_\varepsilon} u^2(t,x)\,dx\leq\varepsilon. \tag{1.16}$$

Soit $N\geq 1$ un entier et soient N réels positifs $0<c_1<\cdots<c_N$. Il existe $\alpha=\alpha(c_1,\dots,c_N,\rho)>0$ tel que, s'il existe N fonctions $x_1,\dots,x_N:\mathbb{R}^+\to\mathbb{R}$ de classe $\mathscr{C}^1$ qui satisfont

$$\forall\,t\geq 0,\qquad \left\|u(t)-\sum_{i=1}^N Q_{c_i}(\cdot-x_i(t))\right\|_{H^1}\leq\alpha, \tag{1.17}$$

et

$$\forall\,t\geq 0,\ \forall\,i\in\{1,\dots,N-1\},\qquad x_{i+1}(t)-x_i(t)\geq|\ln\alpha|, \tag{1.18}$$

alors u est un multi-soliton (en $+\infty$). Autrement dit, il existe $\theta>0$, $0<c_1^+<\cdots<c_N^+$, $x_1^+,\dots,x_N^+\in\mathbb{R}$ et des constantes positives C_s telles que pour tout $s\geq 0$, pour tout $t\geq 0$,

$$\left\|u(t)-\sum_{i=1}^N Q_{c_i^+}(\cdot-c_i^+t-x_i^+)\right\|_{H^s}\leq C_s e^{-\theta t}.$$

Le théorème précédent fournit ainsi une caractérisation dynamique des multi-solitons. Observons que les hypothèses sont faites ici pour des temps positifs $t\geq 0$ et non pas pour tous les temps $t\in\mathbb{R}$. De la sorte, ce théorème étend mais raffine également le théorème 1.11. Bien sûr, dans le cas L^2-sous-critique $1<p<5$, les hypothèses (1.17) et (1.18) peuvent être allégées et ne tenir en somme qu'au temps $t=0$ dans la mesure où les sommes de solitons découplés sont stables [80].

Dans la preuve du théorème 1.13 que nous exposons, deux ingrédients majeurs sont utilisés. D'une part, la propriété (3.6) de non-dispersion permet d'obtenir un contrôle de u et de sa dérivée $\partial_x u$ sur la région $x\leq\beta t$ pour un certain $\beta>0$ suffisamment petit ; cela implique que

$$\|u(t)\|_{H^1(x\leq\beta t)}\to 0\quad\text{lorsque } t\to+\infty.$$

D'autre part, la propriété de stabilité asymptotique dans l'espace d'énergie, dont nous disposons grâce aux hypothèses complémentaires (1.17) et (1.18), permet d'avoir un contrôle, lorsque $t\to+\infty$, d'une quantité de la forme $\|u(t)-\sum_{i=1}^N Q_{c_i^+}(\cdot-\rho_i(t))\|_{H^1(x\geq\beta t)}$ où $0<c_1^+<\cdots<c_N^+$ et où pour tout i, les fonctions $t\mapsto\rho_i(t)\in\mathbb{R}$ de classe $\mathscr{C}^1$ proviennent d'un argument de modulation.
Lorsque les deux propriétés intermédiaires sont réunies, nous obtenons que

$$\left\|u(t)-\sum_{i=1}^N Q_{c_i^+}(\cdot-\rho_i(t))\right\|_{H^1}\to 0,\quad\text{lorsque}\quad t\to+\infty,$$

où $\rho_{i+1}(t) - \rho_i(t) \geq \delta t$ et $\rho_i'(t) \geq \delta$ pour un certain $\delta > 0$. Il reste alors à affiner ce résultat pour obtenir la convergence suivante, pour certains $x_1^+, \dots, x_N^+ \in \mathbb{R}$:

$$\left\| u(t) - \sum_{i=1}^{N} Q_{c_i^+}(\cdot - c_i^+ t - x_i^+) \right\|_{H^1} \to 0, \quad \text{lorsque} \quad t \to +\infty.$$

Pour ce faire, nous suivons le schéma de la preuve de Martel [63, Proposition 4] et montrons en fait que la dernière convergence vers 0 est réalisée à vitesse exponentielle.

Il convient de mentionner que le contexte particulier de (KdV) qui correspond à $p = 2$ offre une caractérisation simplifiée des multi-solitons ; aussi, nous obtenons, à l'aide des résultats de résolution en solitons d'Eckhaus et Schuur [28] et de Schuur [101], le théorème suivant.

Théorème 1.14 (F.). *Soit $u_0 \in \mathscr{S}(\mathbb{R}) \setminus \{0\}$. On suppose que la solution u de (KdV) qui correspond à la donnée initiale u_0, qui est globale en temps, est non-dispersive en temps positif, c'est-à-dire satisfait* (1.16). *Alors u est un multi-soliton.*

Ajoutons que nous pouvons également caractériser de façon analogue les solutions non-dispersives de l'équation (mKdV) : ce sont génériquement des multi-breathers.

L'équation (mKdV) admet des solutions particulières appelées *breathers* qui interviennent également dans la réponse à la question de la conjecture de résolution en solitons. Pour tout $(\alpha, \beta) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$, et tous $x_1, x_2 \in \mathbb{R}$, rappelons que le breather B_{α,β,x_1,x_2}, caractérisé par la vitesse d'enveloppe $\gamma := \beta^2 - 3\alpha^2$, la vitesse de phase $\delta := 3\beta^2 - \alpha^2$ et les paramètres de translation $x_1, x_2 \in \mathbb{R}$, a pour expression :

$$B_{\alpha,\beta,x_1,x_2}(t,x) := 2\sqrt{2}\partial_x \left[\arctan \left(\frac{\beta}{\alpha} \frac{\sin(\alpha(x - \delta t - x_1))}{\cosh(\beta(x - \gamma t - x_2))} \right) \right]. \tag{1.19}$$

On pourra consulter notamment Alejo and Muñoz [1] pour l'introduction et l'étude de ces solutions dans l'espace de Sobolev $H^2(\mathbb{R})$.
Un résultat de décomposition en termes de solitions et breathers pour (mKdV) a été développé dans [101, Chapter 5, Theorem 5.1] et récemment dans [8, Theorem 1.10] pour des données initiales u_0 *génériques* (en un sens que nous préciserons au chapitre 3).

Exploitant ce résultat de décomposition, nous obtenons l'énoncé suivant en ce qui concerne les solutions non-dispersives de (mKdV).

Théorème 1.15 (F.). *Soit $p = 3$, soit $u_0 \in \mathscr{S}(\mathbb{R}) \setminus \{0\}$ générique, telle que la solution globale correspondante u de (mKdV) soit non-dispersive pour les temps positifs (c'est-à-dire satisfasse* (1.16)*).*

Alors u est un multi-breather en $+\infty$ avec des vitesses positives : il existe $N_1, N_2 \in \mathbb{N}$ avec $N_1 + N_2 \geq 1$, il existe $0 < c_1 < \cdots < c_{N_1}$, il existe $\alpha_j, \beta_j \in \mathbb{R}_+^*$ pour $j = 1, \dots, N_2$ avec

$$0 < \beta_1^2 - 3\alpha_1^2 < \cdots < \beta_{N_2}^2 - 3\alpha_{N_2}^2$$

et il existe $\gamma > 0$, des constantes positives C_s, des signes $\epsilon_i = \pm 1$ et des réels $x_{0,i}, x_{1,j}, x_{2,j}$ tels que pour tout $s \geq 0$, u appartienne à $\mathscr{C}([0, +\infty), H^s(\mathbb{R}))$ et

$$\forall\, t \geq 0, \qquad \left\| u(t) - \sum_{i=1}^{N_1} \epsilon_i R_{2c_i, x_{0,i}}(t) - \sum_{j=1}^{N_2} B_{\sqrt{2}\alpha_j, \sqrt{2}\beta_j, x_{1,j}, x_{2,j}}(t) \right\|_{H^s} \leq C_s e^{-\gamma t}.$$

La preuve s'inspire de celle du théorème 1.14 : il s'agit d'écrire la décomposition en solitons et breathers disponible pour $p = 3$ et d'utiliser la régularité, l'unicité et les estimées H^s pour les multi-breathers démontrées par Semenov [102].

1.3.2 Comportement ponctuel des multi-solitons de (gKdV) : résultats et conjecture

Nous considérons à présent la question du comportement ponctuel des multi-solitons, à t fixé. Un tel multi-soliton décroît-il comme une somme de solitons ?

A ce sujet, nous démontrons le théorème suivant.

Théorème 1.16 (F.). *Soient $p > 1$, $0 < c_1 < \cdots < c_N$, $x_1, \ldots, x_N \in \mathbb{R}$ et $u \in \mathscr{C}([T, +\infty), H^1(\mathbb{R}))$ un multi-soliton de (gKdV) associé aux solitons R_{c_i,x_i}, $i = 1, \ldots, N$ (u est unique si $p \leq 5$).*
Soit $\alpha < c_1$ et soit $\beta > c_N$.

Alors il existe $T' > 0$, $\kappa_\alpha > 0$ et $\kappa_{\alpha,\beta} > 0$ tels que pour tout $s \in \mathbb{N}$, il existe $C_s > 0$ tel que pour tout $t \geq T$,
si $x \leq \alpha t$, (décroissance exponentielle à gauche de la première onde solitaire)

$$\left|\partial_x^s u(t,x)\right| \leq C_s e^{-\kappa_\alpha |x-\alpha t|}; \tag{1.20}$$

si $\alpha t \leq x \leq \beta t$, (décroissance exponentielle au voisinage des solitons)

$$\left|\partial_x^s u(t,x)\right| \leq C_s \left(e^{-\kappa_{\alpha,\beta}|x-\alpha t|} + \sum_{i=1}^{N} e^{-\sqrt{c_i}|x-c_i t|} \right). \tag{1.21}$$

Pour tout $s \in \mathbb{N}$ et pour tout $n \in \mathbb{N}$, il existe $C_{s,n} > 0$ tel que pour tout $t \geq T$, pour tout $x > \beta t$, (décroissance polynomiale à droite de la dernière onde solitaire)

$$\left|\partial_x^s u(t,x)\right| \leq \frac{C_{s,n}}{(x-\beta t)^n}. \tag{1.22}$$

De plus, si $u \in L^\infty(\mathbb{R}, H^1(\mathbb{R}))$ (par exemple si $p < 5$), alors il existe $T' > 0$, $\kappa_\beta > 0$ tel que pour tout $s \in \mathbb{N}$, il existe $C_s > 0$ tel que pour tout $t \geq T'$, pour tout $x \geq \beta t$, (décroissance exponentielle à droite du dernier soliton)

$$\left|\partial_x^s u(t,x)\right| \leq C_s e^{-\kappa_\beta |x-\beta t|}. \tag{1.23}$$

La décroissance exponentielle à gauche des multi-solitons de (gKdV) (sur une région de la forme $x \leq \alpha t$) est liée au caractère non-dispersif des multi-solitons sur une telle région et découle ainsi d'un argument de monotonie similaire à celui que nous mettons en œuvre pour démontrer le théorème 1.13.

Sur un intervalle de la forme $[\alpha t, \beta t]$, l'inégalité obtenue est conséquence de la décroissance exponentielle en temps des quantités $\left\|u(t) - \sum_{i=1}^N R_{c_i,x_i}(t)\right\|_{H^s}$, $s \in \mathbb{N}$.

Enfin à droite, sur une région de la forme $x > \beta t$, il semble en revanche plus délicat de montrer que les multi-solitons décroissent à vitesse exponentielle en espace. Ici, l'argument de monotonie ne s'applique plus si le multi-soliton n'est pas supposé global. A partir des estimées de convergence exponentielle en temps pour toutes les normes H^s, nous obtenons une décroissance plus rapide que tout polynôme pour chacune des dérivées du multi-soliton. En particulier, nous obtenons que les multi-solitons appartiennent à l'espace de Schwartz.

Le résultat précédent est issu de la considération d'intégrales de la forme

$$I_{s,n}(t) := \int_{x>\beta t} \left(\partial_x^s u\right)^2 (t,x)(x-\beta t)^n \, dx$$

pour s et n entiers naturels. Par récurrence sur n, nous montrons que si pour tout $s \in \mathbb{N}$, $I_{s,n} < +\infty$, alors pour tout $s \in \mathbb{N}$, $I_{s,n+1} < +\infty$. Précisément, sous l'hypothèse que chacune des quantités $I_{0,n}, \dots, I_{s,n}, I_{s+1,n}$ est finie, nous établissons que $I_{s,n+1}$ est finie. Autrement dit, la convergence des intégrales $I_{s,n}$ est démontrée de proche en proche, par procédé « triangulaire » ; par l'intermédiaire des cases grisées, le tableau ci-dessous permet de visualiser toutes les valeurs des indices s et n à considérer pour montrer que $I_{s+1,n} < +\infty$.

s \ n	1	2	…	n	$n+1$
0					
1					
…					
s					
$s+1$					
$s+2$					
…					
$s+n$					

D'un point de vue technique, cette obtention triangulaire de la décroissance polynomiale de u a pour origine l'apparition d'un terme contenant une dérivée d'ordre $s+1$ en espace sur u lorsqu'on dérive en temps une fonctionnelle de la forme $\int_{\mathbb{R}} \left(\partial_x^s u\right)^2 \phi(t,x) \, dx$, où $\phi(t)$ désigne une certaine fonction « poids ». Ce phénomène est évidemment lié à la structure de l'équation (gKdV).

La décroissance rapide obtenue implique en particulier de la non-dispersion à droite pour le multi-soliton u, ce qui permet d'obtenir (1.23) via de la monotonie dans le cas où u est défini pour tout temps $t \in \mathbb{R}$; c'est notamment le cas dans le cas L^2-sous-critique.

Notons qu'un concept essentiel dans la théorie développée pour (gKdV) réside dans « l'effet régularisant de Kato » (ou *Kato smoothing effect*) [47], et si dans les cas L^2-critique et L^2-surcritique, les multi-solitons devaient également décroître à vitesse exponentielle à droite de la dernière onde solitaire (ce qui est actuellement au stade de conjecture), il est possible que le gain de dérivées en espace traduit par cet effet régularisant soit utile à la preuve correspondante.

1.4 Famille de multi-solitons pour les équations de Klein-Gordon

Nous nous intéressons ici à une autre équation pour laquelle il est possible de considérer des multi-solitons : l'équation de Klein-Gordon non-linéaire qui apparaît en physique quantique des champs

$$\partial_t^2 u = \Delta u - u + |u|^{p-1} u, \qquad \text{(NLKG)}$$

où u est une fonction de $(t,x) \in \mathbb{R} \times \mathbb{R}^d$ à valeurs réelles et $p > 1$.

Cette équation de nature hyperbolique présente des caractéristiques différentes des équations (gKdV) et (NLS) introduites dans les parties précédentes. Par exemple, la norme L^2 de $u(t)$ n'est pas conservée au cours du temps ; en outre, (NLKG) ne dispose guère de propriété d'invariance par scaling.

Néanmoins, les deux quantités remarquables suivantes sont conservées par une solution u de (NLKG) à valeurs dans $H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$:

- l'énergie $\frac{1}{2}\int_{\mathbb{R}^d} \left\{(\partial_t u)^2 + |\nabla u|^2 + u^2 - \frac{2}{p+1}|u|^{p+1}\right\}(t,x)\ dx$
- le moment $\int_{\mathbb{R}^d} \{\partial_t u \nabla u\}(t,x)\ dx$.

On observe aussi que l'ensemble des solutions de (NLKG) est laissé invariant sous l'action de « boosts » lorentziens, c'est-à-dire sous l'action de matrices de la forme

$$\Lambda_\beta := \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta^\top & I_d + \frac{\gamma-1}{|\beta|^2}\beta^\top\beta \end{pmatrix}$$

où $\beta \in \mathbb{R}^d$ est de norme euclidienne $|\beta| < 1$ et $\gamma := \frac{1}{\sqrt{1-|\beta|^2}}$.

Aussi, l'équation (NLKG) admet des solitons, obtenus comme « ground states » boostés à partir d'une solution positive $Q \in H^1(\mathbb{R}^d)$ de l'équation fondamentale

$$\Delta Q - Q + Q^p = 0,$$

de la façon suivante :

$$Q_{\beta,x_0} : (t,x) \mapsto Q\left(pr \circ \Lambda_\beta(t, x - x_0)\right),$$

pour un choix de paramètres $\beta \in \mathbb{R}$, $|\beta| < 1$ et $x_0 \in \mathbb{R}^d$, où $\gamma := \frac{1}{\sqrt{1-|\beta|^2}}$ et pr est la projection canonique $\mathbb{R}^{1+d} \to \mathbb{R}^d$ sur les d dernières coordonnées. Dans le cas unidimensionnel en particulier, le soliton précédent se réécrit :

$$Q_{\beta,x_0} : (t,x) \mapsto Q\left(x - \beta t - x_0\right).$$

Ces solitons sont des objets orbitalement instables, et cela indépendamment de la valeur de p [23, 37] ; cela constitue une différence notable avec les équations de Korteweg-de Vries ou de Schrödinger. Comme il est commode de réécrire (NLKG) comme un système différentiel d'ordre 1 et de travailler avec des vecteurs à deux composantes, on notera dans la suite

$$R_{\beta,x_0}(t,x) := \begin{pmatrix} Q_{\beta,x_0}(t,x) \\ \partial_t Q_{\beta,x_0}(t,x) \end{pmatrix}.$$

La première construction d'un multi-soliton dans le contexte de (NLKG) est due à Côte et Muñoz [20] ; elle repose sur des méthodes usuelles de compacité et d'énergie qui peuvent être mises en pratique grâce à la théorie spectrale des opérateurs linéarisés autour des solitons développée dans cet article. Notons que cette construction a été étendue par Côte et Martel [18] en démontrant l'existence de « multi-bound states » qui sont l'analogue des multi-solitons, mais définis à l'aide d'états excités et non plus simplement à partir de « ground states ».

Focalisés plus particulièrement sur la dynamique en temps long de solutions de l'équation (NLKG) au voisinage d'une somme de solitons, nous démontrons l'existence d'une famille à N paramètres de multi-solitons. Précisément, nous avons

Théorème 1.17 (F.)**.** *Soit $p > 2$. Considérons un entier naturel $N \geq 2$ et $2N$ paramètres*

$$x_1, \dots, x_N \in \mathbb{R}^d \quad \textit{et} \quad \beta_1, \dots, \beta_N \in \mathbb{R}^d$$

tels que $0 < |\beta_N| < \cdots < |\beta_1| < 1$.
Alors il existe $\sigma > 0$, $0 < e_{\beta_1} < \cdots < e_{\beta_N}$, $Y_{+,i} \in \mathscr{C}_b(\mathbb{R}, H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d))$ pour $i = 1, \dots, N$ et une famille à N paramètres $(\varphi_{A_1,\dots,A_N})_{(A_1,\dots,A_N)\in\mathbb{R}^N}$ de solutions de (NLKG) *telle que pour tout $(A_1, \dots, A_N) \in \mathbb{R}^N$, il existe $t_0 \in \mathbb{R}$ et $C > 0$ tels que*

$$\forall\, t \geq t_0, \qquad \left\| \Phi_{A_1,\dots,A_N}(t) - \sum_{i=1}^N R_{\beta_i,x_i}(t) - \sum_{i=1}^N A_i e^{-e_{\beta_i} t} Y_{+,i}(t) \right\|_{H^1\times L^2} \leq C e^{-(e_{\beta_N}+\sigma)t}, \quad (1.24)$$

où $\Phi_{A_1,\dots,A_N} := \begin{pmatrix} \varphi_{A_1,\dots,A_N} \\ \partial_t \varphi_{A_1,\dots,A_N} \end{pmatrix}$. De plus, si $(A'_1, \dots, A'_N) \neq (A_1, \dots, A_N)$, alors $\varphi_{A'_1,\dots,A'_N} \neq \varphi_{A_1,\dots,A_N}$.

La famille de multi-solitons décrite par le théorème précédent est construite en adaptant à (NLKG) la construction effectuée par Combet [11,12] dans le cadre des équations (gKdV) et (NLS). Cette construction s'appuie sur un argument topologique établi originellement dans [19] et est possible grâce à la bonne connaissance des opérateurs linéarisés autour des « ground states ». Nous utilisons en effet les notations introduites par Côte et Muñoz [20], réemployées par Côte et Martel [18], ainsi que les propriétés spectrales démontrées dans leur article.

En ce qui concerne la question de la classification, nous montrons réciproquement le théorème suivant.

Théorème 1.18 (F.)**.** *Sous les mêmes hypothèses que le théorème précédent, si u est une solution de* (NLKG) *telle que*

$$\left\| U(t) - \sum_{i=1}^N R_{\beta_i,x_i}(t) \right\|_{H^1\times L^2} = \mathrm{O}\left(\frac{1}{t^\alpha}\right) \quad \textit{lorsque } t \to +\infty, \qquad (1.25)$$

où $U = \begin{pmatrix} u \\ \partial_t u \end{pmatrix}$ et $\alpha > 3$, alors il existe $A_1, \dots, A_N \in \mathbb{R}$ et $t_0 \in \mathbb{R}$ tels que pour tout $t \geq t_0$, $U(t) = \Phi_{A_1,\dots,A_N}(t)$.

L'unicité de cette famille de multi-solitons en restriction à la classe satisfaisant (1.25) est obtenue grâce à la presque-monotonie d'une fonctionnelle $\mathcal{F}$ bien choisie, inspirée de [76]. L'inégalité essentielle qui permet d'améliorer la vitesse de convergence de

$$Z(t) := U(t) - \sum_{i=1}^N R_{\beta_i,x_i}(t)$$

vers 0 dans $H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$ est de la forme

$$-\mathcal{F}'(t) \leq \frac{\lambda}{t}\mathcal{F}(t) + h\left(t, \|Z(t)\|_{H^1\times L^2}\right),$$

pour un certain paramètre $\lambda \in (1, \alpha - 1)$ et une fonction h telle que $t \mapsto t^\lambda h(t, \|Z(t)\|_{H^1 \times L^2})$ est intégrable au voisinage de $+\infty$ et dont le comportement en temps long est meilleur que simplement quadratique en $\|Z(t)\|_{H^1 \times L^2}$, essentiellement en $o\left(\sup_{t' \geq t} \|Z(t')\|^2_{H^1 \times L^2}\right)$.

Lorsqu'un seul soliton est considéré, nous obtenons une caractérisation complète des solutions qui convergent vers ce soliton lorsque $t \to +\infty$.

Théorème 1.19 (F.)**.** *Soit $\beta \in \mathbb{R}^d$ avec $|\beta| < 1$ et soit $p > 2$.*
Il existe $e_\beta > 0$, $Y_{+,\beta} \in \mathscr{C}(\mathbb{R}, H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d))$ et une famille $(u^A)_{A \in \mathbb{R}}$ à un paramètre de solutions de (NLKG) *telle que pour tout $A \in \mathbb{R}$, il existe $t_0 = t_0(A) \in \mathbb{R}$ tel que pour tout $t \geq t_0$*

$$\left\|U^A(t) - R_\beta(t) - Ae^{-e_\beta t} Y_{+,\beta}(t)\right\|_{H^1 \times L^2} \leq Ce^{-2e_\beta t}, \tag{1.26}$$

où $U^A := \begin{pmatrix} u^A \\ \partial_t u^A \end{pmatrix}$. De plus, si $A \neq A'$, alors $u^A \neq u^{A'}$.
Par ailleurs, si u est une solution de (NLKG) *telle que*

$$\|U(t) - R_\beta(t)\|_{H^1 \times L^2} \to 0, \quad \text{lorsque } t \to +\infty, \tag{1.27}$$

où $U = \begin{pmatrix} u \\ \partial_t u \end{pmatrix}$, alors il existe $A \in \mathbb{R}$ et $t_0 \in \mathbb{R}$ tels que pour tout $t \geq t_0$, $U(t) = U^A(t)$.

L'existence d'une infinité de solutions est générée par la direction d'instabilité due à l'existence de la valeur propre négative (liée à e_β) de l'opérateur linéarisé autour du soliton Q_β. Ce phénomène est naturel pour l'étude de solitons instables.

Si l'énoncé précédent se construit sur le modèle de [10] attaché à (gKdV) et s'inspire de plusieurs résultats de classification obtenus auparavant, comme pour (NLS) cubique tridimensionnelle [26], (NLS) H^1-critique [25] ou encore l'équation des ondes dans le cas H^1-critique [24], notre démonstration relève d'une approche différente, dans la mesure où celle-ci repose pleinement sur la théorie spectrale des opérateurs linéarisés au voisinage des solitons.

La classification établie dans le théorème 1.19 laisse supposer que pour $N \geq 1$ quelconque, les N-solitons construits dans le théorème 1.17 forment exactement l'ensemble des multi-solitons u tels que

$$\left\| U - \sum_{i=1}^N R_{\beta_i, x_i}(t) \right\|_{H^1 \times L^2} \underset{t \to +\infty}{\longrightarrow} 0$$

où $U = \begin{pmatrix} u \\ \partial_t u \end{pmatrix}$. Cependant, cela demeure au stade de conjecture et l'application d'un argument usuel de coercivité ne permet apparemment pas de décrire et d'améliorer de façon évidente la vitesse de décroissance de $\left\|U - \sum_{i=1}^N R_{\beta_i, x_i}(t)\right\|_{H^1 \times L^2}$ vers 0. En outre, il existe une différence profonde entre le cas où un seul soliton est considéré ($N = 1$) et le cas où $N \geq 2$: le recours à un argument de type point fixe, possible dans le cas $N = 1$ et initié par Duyckaerts et Merle [24, 25] ne paraît pas adapté au cadre des multi-solitons.

Malgré tout, notre argument permet de récupérer tous les multi-solitons d'une classe raisonnable, plus large que la classe à décroissance exponentielle en temps. De plus, la démonstration du théorème 1.18 donne l'espoir d'une réponse à la question de l'unicité ou de la classification générale des multi-solitons pour d'autres équations aux dérivées partielles, au moins dans la classe des solutions qui convergent polynomialement en temps vers une somme de solitons, c'est-à-dire au sens de (1.25).

1.5 Organisation des chapitres

Les chapitres qui suivent cette introduction générale sont consacrés aux développements des résultats obtenus dans le cadre de cette thèse.

Aussi, le deuxième chapitre étudie les propriétés de régularité et d'unicité des multi-solitons des équations de Schrödinger non-linéaires (NLS) et s'attache tout particulièrement aux théorèmes 1.9 et 1.10.

Le troisième volet ouvre sur la propriété de non-dispersion qui s'avère essentielle pour caractériser les multi-solitons des équations de Korteweg-de Vries généralisées (gKdV). Ce chapitre a donc vocation à contenir les démonstrations des théorèmes 1.13, 1.14 et 1.15.

La quatrième partie fait en outre état du comportement des multi-solitons de (gKdV) en espace (théorème 1.16).

Enfin, la cinquième partie fait l'objet de la construction de multi-solitons pour les équations de Klein-Gordon non-linéaires (NLKG) avec le théorème 1.17 ; elle aborde aussi le problème d'unicité pour la classe de solutions qui convergent vers un soliton (théorème 1.19) ou vers une somme de solitons (théorème 1.17).

Chapter 2

On smoothness and uniqueness of multi-solitons of the non-linear Schrödinger equations

Abstract
In this paper, we study some properties of multi-solitons for the non-linear Schrödinger equations in $\mathbb{R}^d$ with general non-linearities. Multi-solitons have already been constructed in $H^1(\mathbb{R}^d)$ in [19, 71, 85]. We show here that multi-solitons are smooth, depending on the regularity of the non-linearity. We obtain also a result of uniqueness in some class, either when the ground states are all stable, or in the mass-critical case.

2.1 Introduction

2.1.1 Generalities on the non-linear Schrödinger equations

We consider non-linear Schrödinger equations in $\mathbb{R}^d$ which admit traveling solitary waves (solitons). More precisely, we focus on

$$\partial_t u = i\big(\Delta u + f(|u|^2)u\big), \tag{NLS}$$

where $u : I \times \mathbb{R}^d \to \mathbb{C}$, $I \subset \mathbb{R}$ is a time interval, and $f : [0, +\infty) \to \mathbb{R}$ is an H^1-subcritical non-linearity.

For $d \leq 3$ and for particular functions f, equation (NLS) arises in the mathematical description of many physical phenomena; it is used mainly to model non-linear wave dynamics. For instance, it is fundamental in the description of the dynamic of particles moving in electromagnetic fields [61] and quantum systems like Bose-Einstein condensates [98]. With particular non-linearities obtained by linear combinations of quadratic, cubic, and quintic terms it appears also when one tries to describe the propagation of laser beams in some mediums [4] or of more general ultrashort optical pulses (see for example [22] for the study of some solutions of these equations), with applications in medical imaging, material processing and optical communications (we refer to [29] for further

Ce chapitre fait l'objet d'un article rédigé en collaboration avec Raphaël Côte et accepté pour publication dans Communications in Partial Differential Equations [16].

details concerning the applications of (NLS) to fiber optics for example).

For the purpose of the exposition, we focus in this paragraph on pure power non-linearities

$$f(r) = r^{\frac{p-1}{2}}, \quad 1 < p < 1 + \frac{4}{(d-2)_+}, \quad r \geq 0. \tag{2.1}$$

(If $d = 1$ or 2, the condition is $p > 1$ and if $d \geq 3$, the condition is $1 < p < 1 + \frac{4}{d-2}$). We will give results on general non-linearities in paragraph 2.1.3.

Ginibre and Velo [34] proved that (NLS) is locally well-posed in $H^1(\mathbb{R}^d)$: for all $u_0 \in H^1(\mathbb{R}^d)$, there exist $T > 0$ and a unique maximal solution $u \in \mathscr{C}([0,T), H^1(\mathbb{R}^d))$ of (NLS) such that $u(0) = u_0$. For any such H^1 solution, the following quantities are conserved for all $t \in [0,T)$:

- the L^2 mass $\displaystyle\int_{\mathbb{R}^d} |u(t,x)|^2 \, dx.$
- the energy $\displaystyle\int_{\mathbb{R}^d} \left(\frac{1}{2} |\nabla u(t,x)|^2 - \frac{1}{p+1} |u(t,x)|^{p+1} \right) dx.$
- the momentum $\displaystyle\operatorname{Im} \int_{\mathbb{R}^d} \nabla u(t,x) \overline{u}(t,x) \, dx.$

Furthermore, for all $s \in \mathbb{N} \setminus \{0,1\}$, if

$$g : z \mapsto z f(|z|^2) = z|z|^{p-1}$$

is $\mathscr{C}^s$ on $\mathbb{C}$ as an $\mathbb{R}$-differentiable function (that is if $p > s$ or p is an odd integer), and in case where $s < \frac{d}{2}$, if in addition $p < 1 + \frac{4}{d-2s}$, then (NLS) is locally well-posed in $H^s(\mathbb{R}^d)$ according to Kato [48, Theorem 4.1].

Also (NLS) is invariant under properties of space-time translation, phase, and galilean invariances: if $t^0 \in \mathbb{R}$, $v \in \mathbb{R}^d$, $x^0 \in \mathbb{R}^d$, $\gamma \in \mathbb{R}$, and u is a solution to (NLS), then

$$(t,x) \mapsto u(t - t^0, x - x^0 - vt) e^{i\left(\frac{1}{2} v \cdot x - \frac{|v|^2}{4} t + \gamma\right)} \tag{2.2}$$

is also a solution to (NLS). What is more, (NLS) with a pure power non-linearity (2.1) is scaling invariant: if $\lambda > 0$ and u is a solution to (NLS), then

$$(t,x) \mapsto \frac{1}{\lambda^{1/(p-1)}} u\left(\frac{t}{\lambda}, \frac{x}{\lambda^{1/2}}\right) \tag{2.3}$$

is still a solution to (NLS).

Let us introduce now some particular solutions of (NLS) which are essential in the theory and on which our paper is based. Given $\omega > 0$, Berestycki and Lions [2] proved the existence of a (non-vanishing) positive radial solution $Q_\omega \in H^1(\mathbb{R}^d)$ to the following elliptic problem

$$\Delta Q_\omega + f(Q_\omega^2) Q_\omega = \omega Q_\omega, \quad Q_\omega > 0 \tag{2.4}$$

(By scaling, it suffices to prove the existence for $\omega = 1$). A solution to (2.4) is called a *ground state* (and if one relaxes the sign condition, we speak of *bound state*). Using a Pohozaev identity [99], one can show that (2.4) has no solution in $H^1(\mathbb{R}^d)$ for $p \geq 1 + \frac{4}{(d-2)_+}$. Moreover, for $s \in \mathbb{N}^*$ and if g is $\mathscr{C}^s$ on $[0, +\infty)$, then Q_ω is $\mathscr{C}^{s+2}$ on $\mathbb{R}^d$ and one has exponential decay (see [2, proof of Lemma 1]): there exists $C_s > 0$ such that for each multi-index $\delta \in \mathbb{N}^d$ with $|\delta| \leq s+2$,

$$\forall\, x \in \mathbb{R}^d, \qquad \left|\partial^\delta Q_\omega(x)\right| \leq C_s e^{-\frac{\sqrt{\omega}}{2}|x|}. \tag{2.5}$$

Then the function

$$(t,x) \mapsto Q_\omega(x) e^{i\omega t} \tag{2.6}$$

is a solution to (NLS). Using the invariances (2.2) of the equation, one obtains a whole family of solutions of (NLS) known as *solitons*.

Dynamical properties of solitons have been extensively studied. One important result is related to their orbital stability: solitons are orbitally stable if $p < 1 + \frac{4}{d}$ and unstable if $p \geq 1 + \frac{4}{d}$. Recall that the case $p = 1 + \frac{4}{d}$ corresponds to the L^2-critical exponent: in this particular case, the L^2 norm of a solution is invariant by scaling (2.3).

In this article, we are interested in qualitative properties of multi-solitons, that is solutions of (NLS) which behave as a sum of decoupled solitary waves for large times.

Let us begin with the definition of some further notations. Fix $K \in \mathbb{N} \setminus \{0, 1\}$ and for all $k = 1, \ldots, K$, let

$$\omega_k > 0, \quad \gamma_k \in \mathbb{R}, \quad x_k^0 \in \mathbb{R}^d, \quad \text{and } v_k \in \mathbb{R}^d \text{ such that for all } k \neq k', \quad v_k \neq v_{k'}.$$

For all $k = 1, \ldots, K$, we consider

$$R_k(t,x) = Q_{\omega_k}(x - x_k^0 - v_k t) e^{i\left(\frac{1}{2} v_k \cdot x + \left(\omega_k - \frac{|v_k|^2}{4}\right)t + \gamma_k\right)},$$

which is a soliton of (NLS) moving on the line $x = x_k^0 + v_k t$. We denote also

$$R := \sum_{k=1}^K R_k.$$

In general, R is obviously not a solution to (NLS) because of the non-linearity. A *multi-soliton* is a solution u of (NLS) defined on $[T_0, +\infty)$ for some $T_0 \in \mathbb{R}$ and such that

$$\lim_{t \to +\infty} \|u(t) - R(t)\|_{H^1} = 0. \tag{2.7}$$

Multi-solitons were explicitly constructed in the integrable case, that is with $f(x) = x$ and $d = 1$, using the inverse scattering method (see Zakharov and Shabat [113]).

The first construction in a non-integrable context is due to Merle [85], in the critical case $p = 1 + \frac{4}{d}$. Later, following closely the ideas of Martel in [63] for the construction of multi-solitons for the Korteweg-de Vries equations, Martel and Merle [71] constructed multi-solitons of (NLS), in the L^2-subcritical case $1 < p < 1 + \frac{4}{d}$. This result was extended to L^2-supercritical exponent by Côte, Martel and Merle [19]. Let us recall the results.

Theorem 2.1 (Merle [85], Martel and Merle [71], Côte, Martel and Merle [19])**.** *There exist $\theta > 0$ (depending on v_k, ω_k for $1 \leq k \leq K$), $T_0 \geq 0$, and a solution $u \in \mathscr{C}([T_0, +\infty), H^1(\mathbb{R}^d))$ of* (NLS) *such that*

$$\forall\, t \geq T_0, \qquad \|u(t) - R(t)\|_{H^1} \leq e^{-2\theta t}. \tag{2.8}$$

Let us also mention the works by Le Coz and Tsai [60] and Le Coz, Li and Tsai [59] where *infinite* trains of solitons are constructed, in the context of (NLS). The construction of multi-solitons in H^1 was done for many other non-linear dispersive models (besides the generalized Korteweg-de Vries equations) such as the non-linear Klein-Gordon equation [20], the Hartree equation [52], the water-waves system [88], and in both stable and unstable contexts, which means assuming that all Q_{ω_k} are stable or not.

Even though solutions of (NLS) behaving as a sum of decoupled *general bound states* (that is, solutions to (2.4) which change sign) have been studied in the last years (see for example [17] on (NLS) or [18] on non-linear Klein-Gordon equation), in the present paper we concentrate only on multi-solitons based on ground states. Our goal here is to study uniqueness and smoothness issues.

To our knowledge, the only work where multi-solitons are shown to be more regular than H^1 is for the generalized Korteweg-de Vries equation (which is one-dimensional), with monomial non-linearity, by Martel [63], where the exponential convergence (2.8) is shown to hold in $H^s(\mathbb{R})$ for all $s \in \mathbb{N}$ (with a constant C_s depending on s in front of the exponential term and a convergence rate θ independent of s): see Proposition 5 and its proof for the L^2-subcritical and critical cases; the L^2-supercritical case can be treated likewise, as it is mentioned in [19, Remark 1]).

A natural question is thus to understand for (NLS) whether the multi-soliton u in Theorem 2.1 is smoother than H^1: for example, does it belong to $\mathscr{C}([T_0, +\infty), H^s(\mathbb{R}^d))$ for $s > 1$ and does it hold $\|u(t) - R(t)\|_{H^s} \to 0$ as $t \to +\infty$?

Another natural question is the uniqueness or the classification of multi-solitons. Again, to our knowledge, the only complete study of the question was done for the generalized Korteweg-de Vries equations: multi-solitons were proved to be unique in the L^2-subcritical and critical cases by Martel [63], and were classified in the L^2-supercritical case by Combet [11] (there is a K-parameter family of K-solitons, each instability direction yielding a free parameter). Actually, smoothness of the multi-solitons constructed in Theorem 2.1 is an important ingredient in the proof of uniqueness (or classification) in dimension $d \geq 2$.

2.1.2 Main results

Our first result concerns the construction of a multi-soliton in $H^s(\mathbb{R}^d)$, where the regularity index $s > 1$ depends on the regularity of the function g. We prove in particular that the convergence occurs with an exponential rate in $H^s(\mathbb{R}^d)$. The result is stated here for pure power non-linearities, and we will discuss general non-linearities in the next paragraph.

Theorem 2.2 (Smoothness of multi-solitons)**.** *Assume that $p \geq 3$. Let $\theta > 0$ be defined as in Theorem 2.1 and $s_0 = \lfloor p - 1 \rfloor \geq 2$, or $s_0 = +\infty$ if p is an odd integer. There exist $T_1 > 0$ and $u \in \mathscr{C}([T_1, +\infty), H^{s_0}(\mathbb{R}^d))$ a solution of* (NLS) *with pure power nonlinearity* (2.1) *such that for all non-negative integer $s \leq s_0$, there exists $C_s \geq 1$ such that*

$$\forall\, t \geq T_1, \quad \|u(t) - R(t)\|_{H^s} \leq C_s e^{-\frac{2\theta}{s+1} t}. \tag{2.9}$$

Moreover, if p is an odd integer, then for all integer $s \geq 0$,

$$\forall\, t \geq T_1, \quad \|u(t) - R(t)\|_{H^s} \leq \sqrt{C_s} e^{-\theta t}. \tag{2.10}$$

Remark 2.1. Theorem 2.2 completes Theorem 2.1 by showing the existence of smooth multi-solitons. Notice that its applications are limited to dimensions $d \leq 3$, since we consider the pure power case and due to the H^1-subcritical assumption $p < 1+\frac{4}{d-2}$ which is required for the existence of solitons.

In particular, in dimension $d = 1$ and $d = 2$, multi-solitons belong to $H^\infty(\mathbb{R}^d)$ when p is an odd integer, and in dimension $d = 3$, multi-solitons are $H^\infty(\mathbb{R}^3)$ when $p = 3$ (which corresponds to the most physically relevant case).

The exponential decay rate $\frac{2\theta}{s+1}$ is optimal for $s \geq 2$ (obviously, the estimate (2.9) obtained for $s = 1$ is worse than (2.8) but this is not of interest in the context of Theorem 2.2). We underline that this exponential decay rate does depend on s (vanishing when s is large); this is due to some loss, passing from the proof of the H^s-estimate to that of the H^{s+1}-estimate. This dependence could be a problem for some applications. Observe however that if one is willing to consider only regularity indices $s \leq \frac{s_0}{2}$ (say), then a straightforward interpolation argument between the H^1 and H^{s_0} bounds gives the convergence with uniform exponential decay rate θ:

$$\forall s \leq \frac{s_0}{2},\ \forall\, t \geq T_1, \quad \|u(t) - R(t)\|_{H^s} \leq C'_s e^{-\theta t}.$$

Our second goal is to obtain some kind of uniqueness result for (NLS). We derive one for L^2-subcritical and critical (NLS), in the class of multi-solitons u such that $\|u(t) - R(t)\|_{H^1}$ decreases faster than a high power of $\frac{1}{t}$ for large values of t. More precisely, we state the following:

Theorem 2.3 (Conditional uniqueness)**.** *Let $d \leq 2$ and $3 \leq p \leq 1 + \frac{4}{d}$. There exists $N \in \mathbb{N}$ large such that there is a unique $u \in \mathscr{C}([T_1, +\infty), H^1(\mathbb{R}^d))$ solution to* (NLS) *such that*

$$\|u(t) - R(t)\|_{H^1} = \mathrm{O}\left(\frac{1}{t^N}\right), \qquad \text{as } t \to +\infty. \tag{2.11}$$

In particular, the multi-solitons of Theorems 2.1 and 2.2 coincide (and one can take $T_0 = T_1$).

Remark 2.2. The crucial point in Theorem 2.3 is obviously the uniqueness part. For pure power non-linearities, Theorem 2.3 provides conditional uniqueness in the sense of (2.11), in the L^2-subcritical and critical cases with $p \geq 3$ in dimension 1, and in the L^2-critical case $p = 3$ in dimension 2.

The requirement that the non-linearity be L^2-subcritical or L^2-critical is to be expected as no uniqueness holds in the L^2-supercritical case; see Côte and Le Coz [17] for example.

Of course, one would expect an unconditional uniqueness result, that is uniqueness in the class of solutions u defined for large enough times and convergent to the profile R without decay rate:

$$\|u(t) - R(t)\|_{H^1} \to 0 \quad \text{as} \quad t \to +\infty.$$

This seems out of reach with our method, but we would like to point out that Theorem 2.3 already allows to break the class of exponential convergence, in which multi-solitons naturally lie, and as it was done in [59, 60] for example.

2.1.3 General non-linearities

In order to consider general non-linearities, one must make a number of assumptions which we discuss in this paragraph.

Well-posedness in $H^1(\mathbb{R}^d)$ is classically done under the hypothesis that $g : \mathbb{C} \to \mathbb{C}$ is $\mathscr{C}^1$ and satisfies

(H1) $g(0) = 0$ and there exists $p \in \left(1, 1 + \frac{4}{d-2}\right)$ such that $\frac{\partial g}{\partial x}(z), \frac{\partial g}{\partial y}(z) = O(|z|^{p-1})$ as $|z| \to +\infty$.

In order that the Cauchy problem for (NLS) be well-posed in H^s for $s \in \mathbb{N} \setminus \{0, 1\}$, Kato [48, Theorem 4.1] requires furthermore that g is $\mathscr{C}^s$, and if $s \leq \frac{d}{2}$, one also needs:

(H2) if g is a polynomial in z and $\bar{z}$, its degree is $\deg g < 1 + \frac{4}{d-2s}$
if g is not a polynomial, there exists $p \in \left[\lceil s \rceil, 1 + \frac{4}{d-2s}\right[$ such that $\frac{\partial^s g}{\partial_x^k \partial_y^{s-k}}(z) = O(|z|^{p-s})$ as $|z| \to +\infty$, for all $k = 0, \dots, s$, with $\lceil s \rceil$ the smallest integer greater or equal to s.

The existence of solitons with frequency $\omega > 0$ is not as immediate as in the pure power case. Under the assumption that

$$\exists\, \xi_\omega \in \mathbb{R}_+^*, \qquad F(\xi_\omega) > \omega \xi_\omega \quad \text{where} \quad F(r) := \int_0^r f(\rho)\, d\rho \tag{2.12}$$

for $r \geq 0$, Berestycki and Lions [2] showed the existence of a positive radial ground state $Q_\omega \in H^1(\mathbb{R}^d)$ to (2.4). Note that if there exist $\tilde{\lambda} > 0$, $p' > 0$ and $r_0 > 0$ such that

$$\forall\, r \geq r_0, \quad f(r) \geq \tilde{\lambda} r^{p'},$$

then (2.12) holds for all $\omega > 0$. If $d = 1$, a necessary and sufficient condition for the existence of a positive solution (2.4) is that ω is such that

$$r_0 := \inf \left\{ r > 0 \,\middle|\, F(r) = \omega r \right\} \tag{2.13}$$

exists and $f(r_0) > \omega$ (see [2]).
Let us denote by $\mathcal{O}$ a subset of $(0, +\infty)$ such that

$$\text{for all } \omega \in \mathcal{O}, \text{ there exists a solution } Q_\omega \text{ to (2.4).} \tag{2.14}$$

Recall that when it exists, a positive solution of (2.4) is radial (see [2, section 3] and also Gidas, Ni and Nirenberg [33, Theorem 1'] for non-linearities f such that $r \mapsto r f(r^2)$ is increasing). We underline that it is not unique in general. Indeed, Dàvila, Pino and Guerra [27] showed the existence of at least three positive H^1 solutions of

$$\Delta u + u^p + \lambda u^2 = u$$

for some $\lambda > 0$ and $p \in (1, 5)$ in dimension $d = 3$. See [27] for other counterexamples in dimension 3.
On the other side, Kwong [54] showed uniqueness of a positive radial ground state in the pure power case, and one can extend this to more general non-linearities; we refer to Mc Leod and Serrin [84],

Serrin and Tang [103] and Jang [42] for full details. One of the most important statements may be found in Serrin and Tang [103]: a sufficient condition for uniqueness when $d \geq 3$ is the existence of $\alpha > 0$ such that

$$\begin{cases} \forall\, x \in (0,\alpha], \quad f(x) \leq 1, \quad \text{and} \quad \forall\, x \in (\alpha,+\infty), \quad f(x) > 1 \\ x \mapsto \dfrac{xf'(x)}{f(x)-1} \quad \text{is not increasing on } (\alpha,+\infty). \end{cases}$$

In [42], a slightly more general condition (inspired by [103]) yields uniqueness for (2.4) in any dimension $d \geq 2$.

Let us point out that conditions for existence and uniquenes of a ground state have been discussed for specific non-linearities in the litterature. For example, Berestycki and Lions condition concerning existence and Serrin and Tang condition concerning uniqueness of a ground state apply to the (important) cubic-quintic non-linearity (corresponding to $g(z) = z|z|^2 - z|z|^4$ or $f(r) = r - r^2$). Killip, Oh, Pocovnicu and Visan studied more precisely the properties of ground states associated with this nonlinearity and showed in particular that existence and uniqueness of a positive radially symmetric solution to

$$\Delta Q_\omega + Q_\omega^3 - Q_\omega^5 = \omega Q_\omega$$

hold if and only if $\omega \in \mathcal{O} := \left(0, \frac{3}{16}\right)$; see [51, Lemma 2.1 and Theorem 2.2].

Pursuing with general non-linearities, we will also need a number of assumptions on the linearized operators around solitons. Fix $\omega \in \mathcal{O}$, and let

$$\begin{array}{rccl} \mathscr{L}_\omega : & H^1(\mathbb{R}^d,\mathbb{C}) & \to & H^1(\mathbb{R}^d,\mathbb{C}) \\ & v = v_1 + iv_2 & \mapsto & -\Delta v + \omega v - (f(Q_\omega^2)v + 2Q_\omega^2 f'(Q_\omega^2)v_1) \end{array}$$

so that the linearized equation of (NLS) around $e^{i\omega t}(Q+v)$ is $\partial_t v = i\mathscr{L}_\omega v$. We also define the linearized energy around Q_ω, for any $w = w_1 + iw_2 \in H^1(\mathbb{R}^d,\mathbb{C})$

$$\begin{aligned} H(w) :&= \int_{\mathbb{R}^d} \left(|\nabla w|^2 + \omega |w|^2 - \left(f(Q_\omega^2)|w|^2 + 2Q_\omega^2 f'(Q_\omega^2) w_1^2 \right) \right) \, dx \\ &= \operatorname{Re} \int_{\mathbb{R}^d} \mathscr{L}_\omega w \overline{w} \, dx = \int_{\mathbb{R}^d} L_{+,\omega} w_1 w_1 \, dx + \int_{\mathbb{R}^d} L_{-,\omega} w_2 w_2 \, dx, \end{aligned}$$

where

$$\begin{aligned} L_{+,\omega} w_1 &:= -\Delta w_1 + \omega w_1 - \left(f(Q_\omega^2) + 2Q_\omega^2 f'(Q_\omega^2) \right) w_1 \\ L_{-,\omega} w_2 &:= -\Delta w_2 + \omega w_2 - f(Q_\omega^2) w_2. \end{aligned}$$

We do two (mutually incompatible) coercivity assumptions, depending on whether Q_ω is stable or not. They write as follows:

(H3) (Stable case) There exists $\mu_+ > 0$ such that for all $w = w_1 + iw_2 \in H^1(\mathbb{R}^d,\mathbb{C})$

$$\begin{aligned} H(w) \geq \mu_+ \|w\|_{H^1}^2 - \frac{1}{\mu_+} \left(\int_{\mathbb{R}^d} w_1 Q_\omega \, dx \right)^2 - \frac{1}{\mu_+} \sum_{i=1}^d \left(\int_{\mathbb{R}^d} w_1 \partial_{x_i} Q_\omega \, dx \right)^2 \\ - \frac{1}{\mu_+} \left(\int_{\mathbb{R}^d} w_2 Q_\omega \, dx \right)^2. \end{aligned} \tag{2.15}$$

(H4) (Unstable case) There exists an eigenfunction $Y_\omega = Y_1 + iY_2 \in H^1(\mathbb{R}^d, \mathbb{C})$ of $i\mathscr{L}_\omega$ (with eigenvalue $e_0 > 0$) and $\mu_+ > 0$ such that for all $w = w_1 + iw_2 \in H^1(\mathbb{R}^d, \mathbb{C})$,

$$H(w) \geq \mu_+ \|w\|_{H^1}^2 - \frac{1}{\mu_+}\left(\int_{\mathbb{R}^d} w_1 Y_2 dx\right)^2 - \frac{1}{\mu_+}\sum_{i=1}^d \left(\int_{\mathbb{R}^d} w_1 \partial_{x_i} Q_\omega \, dx\right)^2 - \frac{1}{\mu_+}\left(\int_{\mathbb{R}^d} w_2 Y_1 \, dx\right)^2 + \left(\int_{\mathbb{R}^d} w_2 Q_\omega \, dx\right)^2. \quad (2.16)$$

Assumptions (H3) and (H4) are intimately related to the stability of Q_ω. Regarding the stable case, we have the following result by Grillakis, Shatah and Strauss [38, p. 341-345] (see also the work by Weinstein [108, 109] and by Maris [62, Lemma 2.4]).

Proposition 2.4. *Assume that O is open and that the map $\omega \mapsto Q_\omega$ is of class $\mathscr{C}^1$. Let $\omega_0 \in O$. Under the non-degeneracy assumption that*

$$\operatorname{Ker}(L_{+,\omega_0}) = \operatorname{Span}\left\{\frac{\partial Q_{\omega_0}}{\partial x_i},\ i = 1, \dots, d\right\}, \quad (2.17)$$

we have the following dichotomy:

- *If* $\frac{d}{d\omega}_{|\omega=\omega_0} \int_{\mathbb{R}^d} Q_\omega(x)^2 \, dx > 0$, *then* (2.15) *holds, and as a consequence,* Q_{ω_0} *is orbitally stable in* $H^1(\mathbb{R}^d)$.
- *If* $\frac{d}{d\omega}_{|\omega=\omega_0} \int_{\mathbb{R}^d} Q_\omega(x)^2 \, dx < 0$, *then* Q_{ω_0} *is orbitally unstable in* $H^1(\mathbb{R}^d)$.

We also refer to Cazenave and Lions [5, Theorem II.2 and Remark II.3] for another approach to H^1 orbital stability of the solitons based on Q_{ω_0}.

For the pure power case, $\frac{d}{d\omega}\int_{\mathbb{R}^d} Q_\omega(x)^2 \, dx = \left(\frac{2}{p-1} - \frac{d}{2}\right) \omega^{\frac{-p+3}{p-1} - \frac{d}{2}} \int_{\mathbb{R}^d} Q_1(x)^2 \, dx$ so that it is positive when $1 < p < 1 + \frac{4}{d}$, that is in the L^2-subcritical case (and in particular (H3) holds in that case) and it is negative when $1 + \frac{4}{d} < p < \frac{d+2}{d-2}$, that is in the L^2-supercritical case.

Regarding the unstable case, following the ideas of Duyckaerts and Merle [24], Duyckaerts and Roudenko [26], and Côte, Martel and Merle [19], the coercivity result below holds.

Proposition 2.5 ((3.6) in [19])**.** *Let $\omega \in O$ such that $i\mathscr{L}_\omega$ admits a non zero eigenfunction $Y_\omega \in H^1(\mathbb{R}^d)$. Then* (2.16) *holds.*

An important step is therefore the construction of an eigenfunction Y_ω: this can be done in the L^2-supercritical pure power case ($p > 1 + \frac{4}{d}$) for all $\omega > 0$, and so (H4) holds in that case.

We are now in a position to state our results for general non-linearities. For smoothness, it reads as follows.

Theorem 2.2′. *Let $s_0 > \frac{d}{2}$. Assume that g satisfies (H1) and belongs to $W^{s_0+1,\infty}_{loc}(\mathbb{C})$. Assume moreover that for all $k = 1, \dots, K$, ω_k belongs to O and Q_{ω_k} satisfies either (H3) or (H4). Then the conclusions of Theorem 2.2 hold.*

And below is about uniqueness.

Theorem 2.3′. *Let $d \leq 3$ and $\tilde{f} : z \mapsto f(|z|^2)$ be of class $\mathscr{C}^2$ on $\mathbb{C}$ (as an $\mathbb{R}$-differentiable function), such that its second differential satisfies*

$$\left\|D_z^2 \tilde{f}\right\| = \mathrm{O}\left(|z|^{\frac{4}{d}-2}\right), \qquad \text{as } |z| \to +\infty. \tag{2.18}$$

If f is not the pure power non-linearity, assume that for all $k = 1, \dots, K$, $\omega_k \in \mathcal{O}$ and Q_{ω_k} satisfies (H3), and in the case where $d \geq 2$, assume moreover that g belongs to $W^{s_0+1,\infty}_{loc}(\mathbb{C})$, where $s_0 := \lfloor \frac{d}{2} \rfloor + 1$.
Then the conclusion of Theorem 2.3 holds.

Remark 2.3. Theorems 2.3 and 2.3′ are restricted to dimensions $d \leq 6$. For $d \geq 7$, a similar uniqueness result can be proved (using the same method as that we develop in section 2.3), provided a smaller class of multi-solitons u is considered, and for which a bound on $\|u(t) - R(t)\|_{L^\infty}$ is furthermore assumed. This is the purpose of the next proposition.

Proposition 2.6. *Let $d \geq 4$, $s_0 := \left\lfloor \frac{d}{2} \right\rfloor + 1$, and $\tilde{f} : z \mapsto f(|z|^2)$ be of class $\mathscr{C}^2$ on $\mathbb{C}$ (as an $\mathbb{R}$-differentiable function), such that its second differential satisfies*

$$\left\|D_z^2 \tilde{f}\right\| = \mathrm{O}\left(|z|^{\frac{4}{d}-2}\right), \qquad \text{as } |z| \to +\infty. \tag{2.19}$$

Assume that g belongs to $W^{s_0+1,\infty}_{loc}(\mathbb{C})$. Assume moreover that for all $k = 1, \dots, K$, $\omega_k \in \mathcal{O}$ and Q_{ω_k} satisfies (H3).
Then for any $\alpha > 0$, there exists $N \in \mathbb{N}^*$ such that there exists a unique $u \in \mathscr{C}([T_1, +\infty), H^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d))$ solution to (NLS) *such that*

$$\|u(t) - R(t)\|_{H^1} = \mathrm{O}\left(\frac{1}{t^N}\right) \quad \text{and} \quad \int_t^{+\infty} \|u(s) - R(s)\|_{L^\infty}\, ds = \mathrm{O}\left(\frac{1}{t^\alpha}\right), \qquad \text{as } t \to +\infty.$$

2.1.4 Outline of the paper and the proofs

The main content

We will prove Theorems 2.2′ and 2.3′ which generalize Theorems 2.2 and 2.3 respectively when applied to pure power non-linearities.

Section 2 is devoted to the proof of our regularity result, that is Theorem 2.2′. We start from a well-chosen sequence (u_n) of solutions satisfying uniform H^1 estimates and which were constructed in [19] and [71] (we emphasize that we do not work with the already built multi-soliton in $H^1(\mathbb{R}^d)$ given in Theorem 2.1). Taking some inspiration from Martel [63, section 3] in the context of the generalized Korteweg-de Vries equations, we prove H^s uniform estimates for (u_n) via an induction on the index of regularity. We can combine both stable and unstable cases since we start from the same uniform exponential H^1 estimates obtained in [19] and [71]. From these H^s estimates we deduce (by a usual compactness argument) the existence of a multi-soliton satisfying the conclusions of Theorem 2.2′.

The induction argument relies on the study of a functional related to $\|u_n\|^2_{H^s}$, suitably modified so as to cancel ill-behaved terms; this functional takes the same form in all dimensions (see (2.30) in subsection 2.2.2). In [63], for gKdV, from $s = 3$, all quantities of the type $\|u_n(t) - R(t)\|_{H^s}$ introduced are shown to decrease exponentially in large time with the same rate. Our proof is more technical, insofar as the algebra is not as favorable. In the context of (NLS), the terms involving real and imaginary parts can not be treated in the same way at once, and in dimension $d \geq 2$, derivative can fall on terms in many various ways. As nonlinearities are not necessarily smooth (as it is the case in [63]), we need to count carefully the number of times one can perform an integration by parts. This explains why in the case of (NLS), the rate of the exponential decay on $\|u_n(t) - R(t)\|_{H^s}$ is halved when passing from s to $s+1$ (see (2.25) and Proposition 2.8). We then obtain the decay rate of (2.9) by a simple interpolation argument.

Regarding the regularity assumption on the nonlinearity, recall that the H^1 estimate in Theorem 2.1 holds when $g : z \mapsto zf(|z|^2)$ is of class $\mathscr{C}^1$. As fas as H^{s_0} regularity is concerned, Kato's well-posedness result [48] in $H^{s_0}(\mathbb{R}^d)$ assumes g of class $\mathscr{C}^{s_0}$. In Theorem 2.2′, we require a bit more regularity for g to prove an H^{s_0} estimate for $s_0 \geq 2$ for the multi-soliton. From a technical point of view, many estimates rely indeed on the local boundedness of the derivatives (in the sense of distributions) of the functions $\frac{\partial^s g}{\partial x^{s'} \partial y^{s-s'}}$, where $s = 0, \dots, s_0$ and $s' = 0, \dots, s$. The preceding property is typically used at two levels. First, we need the local Lipschitz condition which is satisfied by functions in $W^{1,\infty}_{loc}$: this is for example the case for (2.42) in subsection 2.2.2. In order to obtain the desired H^{s_0} estimate, we need also to integrate by parts a particular term (at least one time) which contains derivatives with respect to the space variable of maximal order s_0 of both $u_n - R$ and g in order that $u_n - R$ appears with a derivative of order $s_0 - 1$, thus can be controlled (see in particular (2.49) in subsection 2.2.2). For this, one shall ensure that the distributional derivative of $x_i \mapsto \frac{\partial^{s_0} g}{\partial x^s \partial y^{s_0-s}}(R_k(x))$ belong to some Lebesgue space L^q; this is in fact the case if the derivative of $x_i \mapsto \frac{\partial^{s_0} g}{\partial x^s \partial y^{s_0-s}}$ is bounded on a certain disk centered at the origin. Therefore, we assume that g is an element of $W^{s_0+1,\infty}_{loc}(\mathbb{C})$. Notice that this condition is met when f is the pure power nonlinearity (2.1) with $s_0 = \lfloor p-1 \rfloor$ (and also in the particular case when p is an even integer).

Besides, we emphasize that assumption $s_0 > \frac{d}{2}$ in Theorems 2.2 and 2.2′ (which is automatically satisfied for $d \in \{1, 2, 3\}$) seems to be needed to obtain the desired estimates, judging from (2.43). In order to relax this, one should work out an argument involving Strichartz type estimates. But to be effective, the dispersive estimates are to be done on the linearized equation around a sum of solitons, that is a sum of potentials which are decoupled and smooth, but large and not decaying in time. Such estimates would actually be very useful for other purposes, for example the stability of multi-solitons. To our knowledge, they are however not (yet) available.

Section 3 is devoted to the proof of the uniqueness result, which combines some ideas of [63] and of [81]. We will consider a solution satisfying (2.11) and (2.19) and show that it is in fact the multi-soliton constructed in the proof of Theorem 2.2′: we therefore study the difference of these two solutions and show that it is 0. One main tool for this is a Weinstein type functional, which is coercive provided we assume some adequate orthogonality properties. Depending on the stable or L^2-critical case considered, these orthogonality conditions differ. The coercivity result available in the latter case (where $f : r \mapsto r^{\frac{2}{d}}$) is the object of Proposition 2.25. The fact that we do the difference with an already constructed multi-soliton which is sufficiently regular is crucial, at least up to dimension 2. In fact, what we truly need is the $H^2(\mathbb{R}^d)$ decay for $d \geq 2$, and also at several times, that the constructed multi-soliton takes values in $L^\infty(\mathbb{R}^d)$.

Note also that, finding like us his inspiration in [63], Combet [12], in the one-dimensional pure power unstable case, has already obtained estimates similar to those we develop in section 2.3 for general f in the stable case. Last, the lack of (backward in time) monotonicity properties of (NLS) explains somehow the difficulty to obtain unconditional uniqueness, that is to prove uniqueness in the whole class of multi-solitons in the sense of (4.2) (without decay rate); see Remark 2.7 for more details.

Some notations and writing practices used through the text

Solutions of (NLS) or functions constructed with such solutions take values in $\mathbb{C}$. As usual, the modulus of a complex number will be denoted by $|.|$.
Our computations are generally done in all dimensions d. To that extent,

- for any vector $u \in \mathbb{R}^d$, we denote by $\mathrm{Re}(u)$ (respectively $\mathrm{Im}(u)$) the vector of $\mathbb{R}^d$ which components are the real parts (respectively the imaginary parts) of the components of u.
- $\cdot$ denotes the euclidean scalar product in $\mathbb{R}^d$ and $|.|$ denotes also the euclidean norm from which it derives.
- we use the usual notation for multi-indices.

As usual, it is also convenient to denote by C some positive constant which can change from one line to the next but which is always independent of the index of any sequence considered.

The main functional spaces we will work with are the Sobolev spaces $H^s(\mathbb{R}^d)$ for $s \in \mathbb{N}^*$ endowed with the usual norms defined by:

$$\forall\, w \in H^s(\mathbb{R}^d), \quad \|w\|_{H^s} := \left(\sum_{|\alpha| \leq s} \|\partial^\alpha w\|_{L^2}^2 \right)^{\frac{1}{2}}.$$

We consider also $H^\infty(\mathbb{R}^d) := \bigcap_{s \in \mathbb{N}^*} H^s(\mathbb{R}^d)$ and the Sobolev spaces $W^{s,\infty}_{loc}(\mathbb{C})$ (identified with $W^{s,\infty}_{loc}(\mathbb{R}^2)$) for $s \in \mathbb{N}^*$.

Furthermore, many computations are presented formally for ease of reading but can be justified by standard regularization arguments which often involve the local well-posedness of (NLS) in H^s with continuous dependence on compact sets of time (see [21, Theorem 1.6]).

2.2 Existence of smooth multi-solitons of (NLS)

In this section, let us concentrate on the proof of Theorem 2.2′. Let $s_0 > \frac{d}{2}$ be an integer and assume that $g : z \mapsto z f(|z|^2)$ is in $W^{s_0+1,\infty}_{loc}(\mathbb{C})$ and satisfies (H1).

2.2.1 Step 1: Uniform H^1-estimate for a sequence of solutions

In order to prove Theorem 2.2, we start from the following proposition, which applies to both stable and unstable cases, and which has already been established in preceding papers. This proposition gives rise to some control in the H^1 norm on a constructed sequence of solutions of (NLS) which turns out to be relevant to achieve our goal.

Proposition 2.7 (Martel and Merle [71], Côte, Martel and Merle [19])**.** *There exist an increasing sequence $(S_n)_{n\in\mathbb{N}}$ of times such that $S_n \to +\infty$, a sequence $(\phi_n) \in \left(H^{s_0}(\mathbb{R}^d)\right)^{\mathbb{N}}$, and constants $\theta > 0$, $C_1 > 0$, $T_0 > 0$ with $S_0 > T_0$ such that for all $n \in \mathbb{N}$:*

- $\|\phi_n\|_{H^{s_0}} \le C_1 e^{-2\theta S_n}$
- *the maximal solution u_n of* (NLS) *such that*

$$u_n(S_n) = R(S_n) + \phi_n$$

belongs to $\mathscr{C}([T_0, S_n], H^{s_0}(\mathbb{R}^d))$ and satisfies

$$\forall\, t \in [T_0, S_n], \qquad \|u_n(t) - R(t)\|_{H^1} \le C_1 e^{-2\theta t}.$$

Remark 2.4. Note that the sequence (ϕ_n) can be chosen in the following form.

- For the stable case, we take $\phi_n = 0$ for all n (see [71]).
- For the unstable case, we take $\phi_n = i \sum_{k\in\{1,\ldots,K\},\pm} b^{\pm}_{n,k} Y^{\pm}_k(S_n)$ for all n with $Y^{\pm}_k$ defined by

$$Y^{\pm}_k : (t,x) \mapsto Y^{\pm}_{\omega_k}\left(x - v_k t - x^0_k\right) e^{i\left(\frac{1}{2} v_k \cdot x + \left(\omega_k - \frac{|v_k|^2}{4}\right)t + \gamma_k\right)} \tag{2.20}$$

and with $b_n = \left(b^{\pm}_{n,k}\right)_{k\in\{1,\ldots,K\},\pm} \in \mathbb{R}^{2K}$ well chosen (see [19] for full details) so that

$$\forall\, n \in \mathbb{N}, \quad |b_n| \le e^{-2\theta S_n}. \tag{2.21}$$

Some particular estimates will be useful throughout the proof. Firstly we retain

$$\|u_n(S_n) - R(S_n)\|_{H^{s_0}} \le C_1 e^{-2\theta S_n} \tag{2.22}$$

(since (2.21) holds and the quantities $\|Y^{\pm}_k(t)\|_{H^s}$ are independent of t). We emphasize also that

$$\forall\, n \in \mathbb{N},\ \forall\, t \in [T_0, S_n], \quad \|u_n(t) - R(t)\|_{H^1} \le C_1 e^{-2\theta t}. \tag{2.23}$$

In addition, the exponential decay property (2.5) of the ground states Q_{ω_k} and their derivatives lead to the following assertion, which is also crucial to establish many estimates:

$$\forall\, t \in [T_0, +\infty),\ \forall\, k \ne k',\ \forall\, |\alpha_1|, |\alpha_2| \in \{0, \ldots, s_0 + 2\}, \quad \int_{\mathbb{R}^d} \left|\partial^{\alpha_1} R_k . \partial^{\alpha_2} R_{k'}\right|(t)\, dx \le C e^{-2\theta t}. \tag{2.24}$$

2.2.2 Step 2: Proof of uniform H^s-estimates for $u_n - R$, $s = 1, \ldots, s_0$

From now on, let

$$v_n := u_n - R.$$

Step 2.1: Performance of preliminary uniform H^s-estimates

Define $\theta_0 := 2\theta$, $\theta_1 := \theta$, and for all $s \geq 2$,

$$\theta_s := \min\left\{\frac{\theta_{s-1}}{2}, \frac{2\theta}{d+1}\right\}, \tag{2.25}$$

so that $\theta_s = \frac{\theta}{2^{s-2}(d+1)}$ for all $s \geq 2$. We prove the following statement, which is the core of our main existence result.

Proposition 2.8. *There exists $T_1 \geq T_0$ such that for all $s \in \{1, \dots, s_0\}$, there exists $C_s \geq 0$ such that for all $n \in \mathbb{N}$, if $S_n \geq T_1$ then*

$$\forall\, t \in [T_1, S_n], \qquad \|v_n(t)\|_{H^s} \leq C_s e^{-\theta_s t}. \tag{2.26}$$

To prove Proposition 2.8, we resort to a "bootstrap" argument. Recall that for all $s \in \mathbb{N}^*$, there exists $\mu_s \geq 0$ such that

$$\forall\, t \in \mathbb{R}, \qquad \|Y_k^{\pm}(t)\|_{H^s} \leq \mu_s. \tag{2.27}$$

For all n, set

$$S_n^* := \inf\left\{t \geq T_0 \mid \forall\, \tau \in [t, S_n],\ \|v_n(\tau)\|_{H^{s_0}} \leq A_{s_0}\right\},$$

for some constant $A_{s_0} > 2K\mu_{s_0}$. Note that S_n^* indeed exists since $v_n(S_n) = i\sum_{k=1}^{K} b_{n,k}^{\pm} Y_k^{\pm}(S_n)$. Hence for all $n \in \mathbb{N}$, we have

$$\forall\, t \in (S_n^*, S_n], \qquad \|v_n(t)\|_{H^{s_0}} \leq A_{s_0}.$$

Due to the continuity of $v_n : [T_0, S_n] \to H^{s_0}(\mathbb{R}^d)$ in S_n^*, we also have for all $n \in \mathbb{N}$:

$$\forall\, t \in [S_n^*, S_n], \quad \|v_n(t)\|_{H^{s_0}} \leq A_{s_0}. \tag{2.28}$$

We will show that S_n^* can be chosen independently of n and improve the preceding estimate by showing first:

Proposition 2.9. *For all $n \in \mathbb{N}$, for all $s \in \{1, \dots, s_0\}$, for all $t \in [S_n^*, S_n]$,*

$$\|v_n(t)\|_{H^s} \leq Ce^{-\theta_s t}.$$

Proof. We argue by induction. The existence of $C_1 \geq 0$ such that for all $n \in \mathbb{N}$,

$$\forall\, t \in [S_n^*, S_n], \qquad \|v_n(t)\|_{H^1} \leq C_1 e^{-\theta t}$$

is already known. Assume that for some $s \in \{2, \dots, s_0\}$, for all $s' \in \{1, \dots, s-1\}$, there exists $C_{s'} \geq 0$ such that for all $n \in \mathbb{N}$,

$$\forall\, t \in [S_n^*, S_n], \qquad \|v_n(t)\|_{H^{s'}} \leq C_{s'} e^{-\theta_{s'} t}. \tag{2.29}$$

We aim at showing that the same estimate is valid for $s' = s$. For this purpose, let us consider for all $n \in \mathbb{N}$ the functional

$$G_{n,s} : t \mapsto \int_{\mathbb{R}^d} \left\{ \sum_{|\alpha|=s} \binom{s}{\alpha} |\partial^\alpha u_n|^2 - \sum_{|\beta|=s-1} \binom{s-1}{\beta} \mathrm{Re}\left(u_n^2 \left(\partial^\beta \overline{u_n}\right)^2\right) f'(|u_n|^2) \right\}(t)\, dx. \tag{2.30}$$

More precisely we prove, in what follows, how to obtain the following statement, which is essential in the proof of estimate (2.29) corresponding to $s' = s$.

Lemma 2.10. *For all $n \in \mathbb{N}$, and for all $t \in [S_n^*, S_n]$, we have*

$$|G_{n,s}(t) - G_{n,s}(S_n)| \leq Ce^{-\min\{\theta_{s-1}, \frac{4\theta}{d+1}\}t}, \tag{2.31}$$

for some constant C independent of n, t and A_{s_0}.

Remark 2.5. The fundamental reason why it is worth introducing the functional $G_{n,s}$ is that no quadratic term involving $\partial^\alpha u_n$ for $|\alpha| = s$ appears in its first derivative and no term $\partial^{\alpha'} u_n$ with $|\alpha'| > s$ appears either. Thus we manage to control $G'_{n,s}(t)$. Nevertheless, we do not claim that the functional $G_{n,s}$ is the only one that can be used to prove (2.29).

Proof of Lemma 2.10. We will work on the derivative of $G_{n,s}$ and show in fact that

$$\left|G'_{n,s}(t)\right| \leq Ce^{-\min\{\theta_{s-1}, \frac{4\theta}{d+1}\}t}. \tag{2.32}$$

The computations and estimates are established rather in terms of the function g instead of f; by this means, they are considerably less burdensome. Besides, in accordance with Remark 2.6 below, the calculations indicate that (2.32) would still be true for more generalized functions g which satisfy $\operatorname{Im} \partial_x g = \operatorname{Re} \partial_y g$.

Let us introduce also some further notations. For ease of reading, we will write u instead of u_n or $u_n(t)$ and G_s instead of $G_{n,s}$. In addition, we denote by u_1 the real part of u and by u_2 its imaginary part.
Moreover, for all $j \in \{1, \dots, d\}$, we specify as e_j the d-tuple $(0, \dots, 1, \dots, 0)$ for which all components except the j-th one are zero.

We divide the proof of Lemma 2.10 into three steps.

Step 1: Computation of the derivative of G_s.

First of all, observe that for all $(x, y) \in \mathbb{R}^2$

$$\begin{aligned} \partial_x g(x,y) &= 2x(x+iy)f'(x^2+y^2) + f(x^2+y^2) \\ \partial_y g(x,y) &= 2y(x+iy)f'(x^2+y^2) + if(x^2+y^2), \end{aligned} \tag{2.33}$$

so that $\left(\partial_x g + i\partial_y g\right)(z) = 2z^2 f'(|z|^2)$. In particular, G_s can be rewritten in terms of g as follows:

$$G_s(t) = \int_{\mathbb{R}^d} \left\{ \sum_{|\alpha|=s} \binom{s}{\alpha} |\partial^\alpha u|^2 - \frac{1}{2} \sum_{|\beta|=s-1} \binom{s-1}{\beta} \operatorname{Re}\left(\left(\partial_x g(u) + i\partial_y g(u)\right) \left(\partial^\beta \overline{u}\right)^2 \right) \right\} (t)\, dx.$$

Let $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$ be such that $|\alpha| = s$. There exists $l(\alpha) \in \{1, \dots, d\}$ such that

$\alpha_{l(\alpha)} \geq 1$. Then, using the fact that u satisfies (NLS), the following holds true:

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^d} |\partial^\alpha u|^2 \, dx =& -2 \operatorname{Im} \int_{\mathbb{R}^d} \left[\sum_{i=1}^{d} \partial_{x_i}^2 \partial^\alpha u + \partial^\alpha (g(u)) \right] \partial^\alpha \overline{u} \, dx \\ =& -2 \operatorname{Im} \int_{\mathbb{R}^d} \partial^\alpha (g(u)) \, \partial^\alpha \overline{u} \, dx \\ =& -2 \operatorname{Im} \int_{\mathbb{R}^d} \left(\partial_x g(u) \operatorname{Re} (\partial^\alpha u) + \partial_y g(u) \operatorname{Im} (\partial^\alpha u) \right) \partial^\alpha \overline{u} \, dx + I_\alpha \\ =& -2 \int_{\mathbb{R}^d} \operatorname{Im}\left(u^2 (\partial^\alpha \overline{u})^2\right) f'(|u|^2) \, dx + I_\alpha,\end{aligned} \tag{2.34}$$

where

$$I_\alpha = -2 \operatorname{Im} \int_{\mathbb{R}^d} \left(\partial^{\alpha - e_{l(\alpha)}} \left(\partial_x g(u)\right) \partial^{e_{l(\alpha)}} u_1 + \partial^{\alpha - e_{l(\alpha)}} \left(\partial_y g(u)\right) \partial^{e_{l(\alpha)}} u_2 \right) \partial^\alpha \overline{u} \, dx.$$

Due to Faà di Bruno formula, I_α is also a linear combination of the following terms:

$$I_{\alpha,q,r,\tilde{\alpha}_1,\dots,\tilde{\alpha}_q}(u) := \int_{\mathbb{R}^d} \partial^{\tilde{\alpha}_1} u_{j_1} \dots \partial^{\tilde{\alpha}_q} u_{j_q} \operatorname{Im} \left(\frac{\partial^q g}{\partial_x^r \partial_y^{q-r}}(u) \partial^\alpha \overline{u} \right) dx, \tag{2.35}$$

where $q \in \{2, \dots, s\}$, $r \in \{0, \dots, q\}$, $\sum_{i=1}^{q} |\tilde{\alpha}_i| = s$, and for all $i \in \{1, \dots, q\}$, $\tilde{\alpha}_i \geq 1$ and $j_i \in \{1, 2\}$.

Similarly, we have for each multi-index β such that $|\beta| = s - 1$:

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}^d} \frac{1}{2} \operatorname{Re} \left(\left(\partial_x g(u) + i \partial_y g(u)\right) \partial^\beta \overline{u}^2 \right) dx \\ &= \operatorname{Im} \int_{\mathbb{R}^d} \left(\partial_x g(u) + i \partial_y g(u)\right) \left(\sum_{j=1}^{d} \partial^{\beta + 2e_j} \overline{u} + \overline{\partial^\beta (g(u))} \right) \partial^\beta \overline{u} \, dx + J_{1,\beta} \\ &= -\sum_{j=1}^{d} \operatorname{Im} \int_{\mathbb{R}^d} \left(\partial_x g(u) + i \partial_y g(u)\right) \left(\partial^{\beta + e_j} \overline{u}\right)^2 \, dx + J_{1,\beta} + J_{2,\beta} + J_{3,\beta} \\ &= -2 \sum_{j=1}^{d} \int_{\mathbb{R}^d} \operatorname{Im}\left(u^2 (\partial^{\beta + e_j} \overline{u})^2\right) f'(|u|^2) \, dx + J_{1,\beta} + J_{2,\beta} + J_{3,\beta},\end{aligned} \tag{2.36}$$

where we denote

$$\begin{aligned}J_{1,\beta} =& \frac{1}{2} \int_{\mathbb{R}^d} \operatorname{Re} \left[\left(\frac{\partial^2 g}{\partial x^2}(u) \partial_t u_1 + \frac{\partial^2 g}{\partial x \partial y}(u) \partial_t u_2 + i \frac{\partial^2 g}{\partial x \partial y}(u) \partial_t u_1 + i \frac{\partial^2 g}{\partial y^2}(u) \partial_t u_2 \right) \left(\partial^\beta \overline{u}\right)^2 \right] dx \\ J_{2,\beta} =& \sum_{j=1}^{d} \operatorname{Im} \int_{\mathbb{R}^d} \partial_{x_j} \left(\frac{\partial g}{\partial x}(u) + i \frac{\partial g}{\partial y}(u) \right) \partial^{\beta + e_j} \overline{u} \partial^\beta \overline{u} \, dx \\ J_{3,\beta} =& \operatorname{Im} \int_{\mathbb{R}^d} \left(\partial_x g(u) + i \partial_y g(u)\right) \overline{\partial^\beta (g(u))} \partial^\beta \overline{u}.\end{aligned}$$

We observe that

$$\sum_{|\alpha|=s}\binom{s}{\alpha}(\partial^\alpha\overline{u})^2-\sum_{|\beta|=s-1}\binom{s-1}{\beta}\sum_{j=1}^d\left(\partial^{\beta+e_j}\overline{u}\right)^2=0. \tag{2.37}$$

Thus,

$$G_s'(t)=\sum_{|\alpha|=s}\binom{s}{\alpha}I_\alpha-\sum_{|\beta|=s-1}\binom{s-1}{\beta}\left(J_{1,\beta}+J_{2,\beta}+J_{3,\beta}\right). \tag{2.38}$$

Remark 2.6. Note that, considering (2.34), (2.36), and (2.37), the property that allows us to obtain (2.38) is in fact Im $\partial_x g$ = Re $\partial_y g$. Indeed, this assumption suffices to have: for all $\alpha\in\mathbb{N}^d$ with $|\alpha|=s$,

$$2\,\mathrm{Im}\left[\left(\partial_x g(u)\mathrm{Re}\,(\partial^\alpha u)+\partial_y g(u)\mathrm{Im}\,(\partial^\alpha u)\right)\partial^\alpha\overline{u}\right]=\mathrm{Im}\left[\left(\partial_x g(u)+i\partial_y g(u)\right)(\partial^\alpha\overline{u})^2\right].$$

Step 2: Control of the derivative of G_s.

Take $\alpha\in\mathbb{N}^d$ such that $|\alpha|=s$. Let q, r, and $\tilde\alpha_1,\dots,\tilde\alpha_q$ be as in (2.35), and denote by $I_{\alpha,q,r,\tilde\alpha_1,\dots,\tilde\alpha_q}(R_k)$ the integral defined exactly as in (2.35) by replacing u by the soliton R_k, for all $k=1,\dots,K$. Then we have

$$\left|I_{\alpha,q,r,\tilde\alpha_1,\dots,\tilde\alpha_q}(u)-\sum_{k=1}^K I_{\alpha,q,r,\tilde\alpha_1,\dots,\tilde\alpha_q}(R_k)\right|\le Ce^{-\min\{\theta_{s-1},\frac{4\theta}{d+1}\}t}, \tag{2.39}$$

for some constant C independent of A_s.

In order to prove (2.39), one proceeds by decomposition of $I_{\alpha,q,r,\tilde\alpha_1,\dots,\tilde\alpha_q}(u)$ as follows. The basic idea is to make terms in v (which provide the expected exponential term at the right-hand side of (2.39)) appear. Let us explicit this decomposition:

$$\begin{aligned}
I_{\alpha,q,r,\tilde\alpha_1,\dots,\tilde\alpha_q}(u)=&\int_{\mathbb{R}^d}\partial^{\tilde\alpha_1}u_{j_1}\dots\partial^{\tilde\alpha_q}u_{j_q}\mathrm{Im}\left(\left(\frac{\partial^q g}{\partial_x^r\partial_y^{q-r}}(u)-\frac{\partial^q g}{\partial_x^r\partial_y^{q-r}}(R)\right)\partial^\alpha\overline{u}\right)dx && (I_{\alpha,1})\\
&+\int_{\mathbb{R}^d}\partial^{\tilde\alpha_1}v_{j_1}\partial^{\tilde\alpha_2}u_{j_2}\dots\partial^{\tilde\alpha_q}u_{j_q}\mathrm{Im}\left(\frac{\partial^q g}{\partial_x^r\partial_y^{q-r}}(R)\partial^\alpha\overline{u}\right)dx && (I_{\alpha,2})\\
&+\int_{\mathbb{R}^d}\partial^{\tilde\alpha_1}R_{j_1}\partial^{\tilde\alpha_2}v_{j_2}\partial^{\tilde\alpha_3}u_{j_3}\dots\partial^{\tilde\alpha_q}u_{j_q}\mathrm{Im}\left(\frac{\partial^q g}{\partial_x^r\partial_y^{q-r}}(R)\partial^\alpha\overline{u}\right)dx && (I_{\alpha,3})\\
&+\dots && (I_{\alpha,\dots})\\
&+\int_{\mathbb{R}^d}\partial^{\tilde\alpha_1}R_{j_1}\partial^{\tilde\alpha_2}R_{j_2}\dots\partial^{\tilde\alpha_{q-1}}R_{j_{q-1}}\partial^{\tilde\alpha_q}v_{j_q}\mathrm{Im}\left(\frac{\partial^q g}{\partial_x^r\partial_y^{q-r}}(R)\partial^\alpha\overline{u}\right)dx && (I_{\alpha,q+1})\\
&+\int_{\mathbb{R}^d}\partial^{\tilde\alpha_1}R_{j_1}\dots\partial^{\tilde\alpha_q}R_{j_q}\mathrm{Im}\left(\frac{\partial^q g}{\partial_x^r\partial_y^{q-r}}(R)\partial^\alpha\overline{v}\right)dx && (I_{\alpha,q+2})\\
&+\int_{\mathbb{R}^d}\partial^{\tilde\alpha_1}R_{j_1}\dots\partial^{\tilde\alpha_q}R_{j_q}\mathrm{Im}\left(\frac{\partial^q g}{\partial_x^r\partial_y^{q-r}}(R)\partial^\alpha\overline{R}\right)dx && (I_{\alpha,q+3})
\end{aligned}$$

Now, we control each preceding term $I_{\alpha,1}, \dots, I_{\alpha,q+3}$ occurring in the preceding decomposition by means of the induction assumption and some classical tools in functional analysis, namely Hölder inequality, Sobolev embeddings, and Gagliardo-Nirenberg inequalities.

Let us notice that

$$\sup_{t\in\mathbb{R}} \|R(t)\|_{L^\infty} < +\infty. \tag{2.40}$$

Considering that $s_0 > \frac{d}{2}$, we deduce then from (2.28) and the Sobolev embedding $H^{\lfloor \frac{d}{2} \rfloor+1}(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d)$ that there exists $A \geq 0$ such that for all n,

$$\forall\, t \in [S_n^*, S_n], \qquad \|u_n(t)\|_{L^\infty} \leq A. \tag{2.41}$$

Since $\frac{\partial^q g}{\partial_x^r \partial_y^{q-r}}$ is $W^{1,\infty}_{loc}$ on $\mathbb{C}$ (or in other words locally Lipschitz on $\mathbb{C}$), it results from (2.40) and (2.41) that

$$|I_{\alpha,1}| \leq C \int_{\mathbb{R}^d} |v||\partial^\alpha u||\partial^{\tilde{\alpha}_1} u| \dots |\partial^{\tilde{\alpha}_q} u|\, dx. \tag{2.42}$$

We now estimate the integral $\int_{\mathbb{R}^d} |v||\partial^\alpha u||\partial^{\tilde{\alpha}_1} u| \dots |\partial^{\tilde{\alpha}_q} u|\, dx$ by means of Hölder inequality. For this, we have to be careful concerning the choice of the involved Lebesgue spaces (or in other words the Hölder exponents) considering that $\partial^{\tilde{\alpha}_i} u \in H^{s_0 - |\tilde{\alpha}_i|}$.
We define $\mathcal{I} := \left\{i \in \{1,\dots,q\} \mid s_0 - |\tilde{\alpha}_i| < \frac{d}{2}\right\}$, $\mathcal{J} := \left\{i \in \{1,\dots,q\} \mid s_0 - |\tilde{\alpha}_i| > \frac{d}{2}\right\}$, and

$$m_i := \begin{cases} \dfrac{2d}{d - 2(s_0 - |\tilde{\alpha}_i|)} & \text{if } i \in \mathcal{I} \\ \infty & \text{if } i \in \mathcal{J}. \end{cases}$$

For $i \in \{1,\dots,q\} \setminus (\mathcal{I} \cup \mathcal{J})$, we take $m_i \in (0,+\infty)$ large enough so that

$$\sum_{i=1}^q \frac{1}{m_i} < \frac{1}{2},$$

which is possible since

$$\begin{aligned} \frac{1}{2} - \left(\sum_{i\in\mathcal{I}} \frac{1}{m_i} + \sum_{i\in\mathcal{J}} \frac{1}{m_i}\right) &= \frac{1}{2} - \sum_{i\in\mathcal{I}} \frac{1}{m_i} = \frac{1}{2} - \sum_{i\in\mathcal{I}} \frac{d - 2(s_0 - |\tilde{\alpha}_i|)}{2d} \\ &\geq \frac{1}{2} - q\left(\frac{1}{2} - \frac{s_0}{d}\right) - \frac{s}{d} \geq (1-q)\left(\frac{1}{2} - \frac{s_0}{d}\right) > 0. \end{aligned} \tag{2.43}$$

due to our assumption on s_0 and the fact that $q > 1$. Then, we observe that for all $i = 1,\dots,q$, $\partial^{\tilde{\alpha}_i} u \in H^{s_0 - |\tilde{\alpha}_i|}(\mathbb{R}^d) \hookrightarrow L^{m_i}(\mathbb{R}^d)$ by the classical Sobolev embedding theorem. Using Hölder inequality, we obtain

$$|I_{\alpha,1}| \leq C\|\partial^\alpha u\|_{L^2} \prod_{i=2}^q \|\partial^{\tilde{\alpha}_i} u\|_{L^{m_i}} \|v\|_{L^m} \leq C\|v\|_{L^m}. \tag{2.44}$$

where $m \geq 2$, and $\frac{1}{m} = \frac{1}{2} - \sum_{i=1}^q \frac{1}{m_i} \geq (q-1)\left(\frac{s_0}{d} - \frac{1}{2}\right) > 0$ by definition of the m_i, $i = 1,\dots,q$. The following Gagliardo-Nirenberg inequality

$$\|v\|_{L^m} \leq C\|v\|^\sigma_{H^{s_0'}} \|v\|^{1-\sigma}_{L^2}, \tag{2.45}$$

with $s_0' := \lfloor \frac{d}{2} \rfloor + 1 \leq \frac{d+1}{2}$ and $\sigma := \frac{d}{s_0'}\left(\frac{1}{2} - \frac{1}{m}\right)$ (which implies $1 - \sigma \geq \frac{1}{s_0'}$, since $2s_0' - d \geq 1$) leads finally to

$$|I_{\alpha,1}| \leq C\|v\|_{L^2}^{1-\sigma} \leq Ce^{-\frac{2\theta}{s_0'}t} \leq Ce^{-\frac{4\theta}{d+1}t}. \tag{2.46}$$

To estimate $I_{\alpha,2}, \dots, I_{\alpha,q+1}$, one proceeds as before. For instance, let us explain how to deal with $I_{\alpha,2}$; the same would be done for the other integrals.
We choose m_1' such that $H^{s_1-1-|\tilde{\alpha}_1|} \hookrightarrow L^{m_1'}(\mathbb{R}^d)$ and $\frac{1}{2} - \frac{1}{m_1'} - \sum_{i=2}^q \frac{1}{m_i} > 0$. Then, again due to Hölder inequality, we have:

$$\begin{aligned} |I_{\alpha,2}| &\leq \|\partial^\alpha u\|_{L^2} \|\partial^{\tilde{\alpha}_1} v\|_{L^{m_1'}} \prod_{i=1}^q \|\partial^{\tilde{\alpha}_i} u\|_{L^{m_i}} \left\| \frac{\partial^q g}{\partial_x^r \partial_y^{q-r}}(R) \right\|_{L^\infty} \\ &\leq C\|v\|_{H^{s-1}} \leq Ce^{-\theta_{s-1}t}. \end{aligned} \tag{2.47}$$

Similarly, we check that

$$\forall\, i \in \{3, \dots, q+1\}, \quad |I_{\alpha,i}| \leq Ce^{-\theta_{s-1}t}. \tag{2.48}$$

Now, let us deal with $I_{\alpha,q+2}$. By (2.24) and the fact that $\frac{\partial^q g}{\partial_x^r \partial_y^{q-r}} \in W^{1,\infty}_{loc}(\mathbb{C})$, we have:

$$I_{\alpha,q+2} = \sum_{k=1}^K \int_{\mathbb{R}^d} \partial^{\tilde{\alpha}_1} R_{k,j_1} \dots \partial^{\tilde{\alpha}_q} R_{k,j_q} \mathrm{Im}\left(\frac{\partial^q g}{\partial_x^r \partial_y^{q-r}}(R_k) \partial^\alpha \overline{v} \right) dx + Ce^{-2\theta t}\|v\|_{H^s}.$$

Again by assumption, each partial derivative of $\frac{\partial^q g}{\partial x^r \partial y^{q-r}}(R_k)$ is bounded, thus the integral

$$\int_{\mathbb{R}^d} \partial^{\alpha - e_l} \overline{v} \partial^{e_l} \left(\partial^{\tilde{\alpha}_1} R_{k,j_1} \dots \partial^{\tilde{\alpha}_q} R_{k,j_q} \frac{\partial^q g}{\partial_x^r \partial_y^{q-r}}(R_k) \right) dx$$

makes sense and one can integrate once by parts to obtain

$$\begin{aligned} &\left| \int_{\mathbb{R}^d} \partial^{\tilde{\alpha}_1} R_{k,j_1} \dots \partial^{\tilde{\alpha}_q} R_{k,j_q} \mathrm{Im}\left(\frac{\partial^q g}{\partial_x^r \partial_y^{q-r}}(R_k) \partial^\alpha \overline{v} \right) dx \right| \\ &\qquad = \left| \mathrm{Im} \int_{\mathbb{R}^d} \partial^{\alpha - e_l} \overline{v} \partial^{e_l} \left(\partial^{\tilde{\alpha}_1} R_{k,j_1} \dots \partial^{\tilde{\alpha}_q} R_{k,j_q} \frac{\partial^q g}{\partial_x^r \partial_y^{q-r}}(R_k) \right) dx \right| \\ &\qquad \leq C\|v(t)\|_{H^{s-1}} \leq Ce^{-\theta_{s-1}t}. \end{aligned} \tag{2.49}$$

Thus,

$$|I_{\alpha,q+2}| \leq Ce^{-\theta_{s-1}t}. \tag{2.50}$$

Finally, by (2.24) and using once more that $\frac{\partial^q g}{\partial_x^r \partial_y^{q-r}}$ is in $W^{1,\infty}_{loc}(\mathbb{C})$,

$$\left| I_{\alpha,q+3} - \sum_{k=1}^K I_{\alpha,q,r,\tilde{\alpha}_1,\dots,\tilde{\alpha}_q}(R_k) \right| \leq Ce^{-2\theta t}. \tag{2.51}$$

Hence, we conclude from (2.46), (2.47), (2.48), (2.50), and (2.51) that (2.39) holds true.

The expressions $J_{1,\beta}, J_{2,\beta}, J_{3,\beta}$ (given before) consist of terms that can be controlled in a similar manner. Let us denote by $J_{i,\beta}(R_k)$ the same integral as $J_{i,\beta}$ where R_k replaces u for all $i = 1, 2, 3$ and for all $k = 1, \dots, K$. One can check that

$$\left| J_{1,\beta} - \sum_{k=1}^{K} J_{1,\beta}(R_k) \right| \leq C e^{-\min\left\{\theta_{s-1}, \frac{4\theta}{d+1}\right\} t}; \tag{2.52}$$

$$\left| J_{2,\beta} - \sum_{k=1}^{K} J_{2,\beta}(R_k) \right| \leq C e^{-\min\left\{\theta_{s-1}, \frac{4\theta}{d+1}\right\} t}; \tag{2.53}$$

$$\left| J_{3,\beta} - \sum_{k=1}^{K} J_{3,\beta}(R_k) \right| \leq C e^{-\theta_{s-1} t}. \tag{2.54}$$

Step 3: Related functional involving R_k.

Let $k \in \{1, \dots, K\}$. An immediate induction argument shows that for all multi-index $\alpha \in \mathbb{N}^d$ such that $|\alpha| = s$, for all multi-index $\alpha' \leq \alpha$, there exists $z_{\alpha'} \in \mathbb{C}$ such that

$$\partial^\alpha R_k(t,x) = \sum_{\alpha' \leq \alpha} z_{\alpha'} \partial^{\alpha'} Q_{\omega_k}(x - x_k^0 - v_k t) e^{i\left(\frac{1}{2} v_k \cdot x + \left(\omega_k - \frac{|v_k|^2}{4}\right) t + \gamma_k\right)}.$$

Therefore

$$\int_{\mathbb{R}^d} |\partial^\alpha R_k|^2 (t)\, dx = \int_{\mathbb{R}^d} \left| \sum_{\alpha' \leq \alpha} z_{\alpha'} \partial^{\alpha'} Q_{\omega_k}(x - x_k^0 - v_k t) \right|^2 dx = \int_{\mathbb{R}^d} \left| \sum_{\alpha' \leq \alpha} z_{\alpha'} \partial^{\alpha'} Q_{\omega_k}(x) \right|^2 dx,$$

so that

$$\frac{d}{dt} \int_{\mathbb{R}^d} |\partial^\alpha R_k|^2 \, dx = 0. \tag{2.55}$$

Furthermore for all multi-index β such that $|\beta| = s - 1$,

$$R_k^2 \left(\partial^\beta \overline{R_k}\right)^2 (t,x) = Q_{\omega_k}^2(x - x_k^0 - v_k t) \left(\sum_{\beta' \leq \beta} z_{\beta'} \partial^{\beta'} Q_{\omega_k}(x - x_k^0 - v_k t) \right)^2,$$

from which we infer also

$$\frac{d}{dt} \int_{\mathbb{R}^d} \mathrm{Re}\big(R_k^2 (\partial^\beta \overline{R_k})\big)^2 f'(|R_k|^2)\, dx = 0. \tag{2.56}$$

Hence, gathering (2.55) and (2.56),

$$0 = \frac{d}{dt} \int_{\mathbb{R}^d} \left\{ \sum_{|\alpha|=s} \binom{s}{\alpha} |\partial^\alpha R_k|^2 - \sum_{|\beta|=s-1} \binom{s-1}{\beta} \mathrm{Re}\left(R_k^2 \left(\partial^\beta \overline{R_k}\right)^2 \right) f'(|R_k|^2) \right\} dx. \tag{2.57}$$

Considering that this last quantity can be written as exactly the same linear combination of terms as $G'_{n,s}$ (we refer to (2.38)) where we replace just u by R_k, we conclude from (2.39), (2.52), (2.53), (2.54), and (2.57) that

$$\forall\, t \in [S_n^*, S_n], \qquad |G'_{n,s}(t)| \leq C e^{-\min\left\{\theta_{s-1}, \frac{4\theta}{d+1}\right\} t}. \tag{2.58}$$

Integrating the preceding inequality between t and S_n yields directly Lemma 2.10. □

Let us now conclude the proof of Proposition 2.9.

We observe that, for all $t \in [S_n^*, S_n]$,

$$
\begin{aligned}
&\sum_{|\alpha|=s} \binom{s}{\alpha} \int_{\mathbb{R}^d} |\partial^\alpha v_n(t)|^2 \, dx \\
&\quad = \left(G_{n,s}(t) - G_{n,s}(S_n)\right) + 2 \sum_{|\alpha|=s} \binom{s}{\alpha} \mathrm{Re} \int_{\mathbb{R}^d} \partial^{\alpha - e_{l(\alpha)}} v_n(t) \partial^{\alpha + e_{l(\alpha)}} \overline{R}(t) \, dx \\
&\quad + \sum_{|\alpha|=s} \binom{s}{\alpha} \left(\int_{\mathbb{R}^d} |\partial^\alpha R(S_n)|^2 \, dx - \int_{\mathbb{R}^d} |\partial^\alpha R(t)|^2 \, dx \right) \\
&\quad + \sum_{|\beta|=s-1} \binom{s-1}{\beta} \left(\int_{\mathbb{R}^d} \mathrm{Re} \left(u_n^2 (\partial^\beta \overline{u_n})^2 \right) f'(|u_n|^2)(t) - \int_{\mathbb{R}^d} \mathrm{Re} \left(R^2 (\partial^\beta \overline{R})^2 \right) f'(|R|^2)(t) \right) \\
&\quad + \sum_{|\beta|=s-1} \binom{s-1}{\beta} \left(\int_{\mathbb{R}^d} \mathrm{Re} \left(R^2 (\partial^\beta \overline{R})^2 \right) f'(|R|^2)(t) - \int_{\mathbb{R}^d} \mathrm{Re} \left(R^2 (\partial^\beta \overline{R})^2 \right) f'(|R|^2)(S_n) \right) \\
&\quad + \sum_{|\alpha|=s} \binom{s}{\alpha} \int_{\mathbb{R}^d} |\partial^\alpha v_n(S_n)|^2 - 2 \sum_{|\alpha|=s} \binom{s}{\alpha} \mathrm{Re} \int_{\mathbb{R}} \partial^{\alpha - e_{l(\alpha)}} v_n(S_n) \partial^{\alpha + e_{l(\alpha)}} \overline{R}(S_n) \, dx.
\end{aligned}
\tag{2.59}
$$

Then, by means of (2.22), (2.55), and (2.56), we infer

$$
\begin{aligned}
\left| \sum_{|\alpha|=s} \binom{s}{\alpha} \int_{\mathbb{R}^d} |\partial^\alpha v_n(t)|^2 \, dx \right| &\leq \left| G_{n,s}(t) - G_{n,s}(S_n) \right| + C \|v_n\|_{H^{s-1}} + C e^{-2\theta t} \\
&+ C \left| \int_{\mathbb{R}^d} \mathrm{Re} \left(u_n^2 (\partial^\beta \overline{u_n})^2 \right) f'(|u_n|^2)(t) \, dx - \int_{\mathbb{R}^d} \mathrm{Re} \left(R^2 (\partial^\beta \overline{R})^2 \right) f'(|R|^2)(t) \, dx \right|.
\end{aligned}
\tag{2.60}
$$

Now, from (2.24), (2.31), (2.57), and from the inequality

$$
\left| \int_{\mathbb{R}^d} \mathrm{Re}\big(u_n^2 (\partial^\beta \overline{u_n})^2\big) f'(|u_n|^2)(t) \, dx - \int_{\mathbb{R}^d} \mathrm{Re}\big(R^2 (\partial^\beta \overline{R})^2\big) f'(|R|^2)(t) \, dx \right| \leq C \|v_n(t)\|_{H^{s-1}},
\tag{2.61}
$$

resulting from the local boundedness of the distributional derivative of $z \mapsto \frac{1}{2}\left(\partial_x g + i \partial_y g\right)(z) = z^2 f'(|z|^2)$, we deduce the existence of $C_s \geq 0$ such that for all $n \in \mathbb{N}$,

$$
\forall\, t \in [S_n^*, S_n], \qquad \|v_n(t)\|_{H^s}^2 \leq C_s^2 e^{-\min\left\{\theta_{s-1}, \frac{4\theta}{d+1}\right\} t}.
\tag{2.62}
$$

This is rewritten as follows:

$$
\forall\, t \in [S_n^*, S_n], \qquad \|v_n(t)\|_{H^s} \leq C_s e^{-\theta_s t},
\tag{2.63}
$$

which is exactly the expected result. Thus the induction argument implies that for all $n \in \mathbb{N}$,

$$
\forall\, t \in [S_n^*, S_n], \qquad \|v_n(t)\|_{H^{s_0}} \leq C_{s_0} e^{-\theta_{s_0} t}.
\tag{2.64}
$$

This puts an end to the proof of Proposition 2.9. □

Now we explain how to deduce from Proposition 2.9 that S_n^* can be chosen independently of n, and by this means, we finish the proof of Proposition 2.8. We pick up $T_1 \geq T_0$ such that $C_{s_0} e^{-\theta_{s_0} T_1} < A_{s_0}$. Let $n \in \mathbb{N}$ be such that $S_n \geq T_1$, and assume by contradiction that $S_n^* > T_1$. Then by continuity of v_n in S_n^* (and by definition of S_n^* as infimum), we have $\|v_n(S_n^*)\|_{H^{s_0}} = A_{s_0}$. On the other hand,

$$\|v_n(S_n^*)\|_{H^{s_0}} \leq C_{s_0} e^{-\theta_{s_0} S_n^*} \leq C_{s_0} e^{-\theta_{s_0} T_1} < A_{s_0}, \tag{2.65}$$

which yields a contradiction. Thus $S_n^* \leq T_1$. Hence, for all $n \in \mathbb{N}$ such that $S_n \geq T_1$, we have

$$\forall\, t \in [T_1, S_n], \qquad \|v_n(t)\|_{H^{s_0}} \leq C_{s_0} e^{-\theta_{s_0} t}.$$

If necessary we drop the first terms of the sequence (S_n) and re-index it in order to obtain:

$$\forall\, n \in \mathbb{N},\ \forall\, t \in [T_1, S_n], \qquad \|v_n(t)\|_{H^{s_0}} \leq C_{s_0} e^{-\theta_{s_0} t}. \tag{2.66}$$

Hence, Proposition 2.8 is established.

Step 2.2: Independence of T_1 with respect to s

Now, we justify that T_1 can be chosen independent of $s > \frac{d}{2}$, which is useful to obtain

$$\forall\, s \in \mathbb{N}^*,\ \exists\, C_s \geq 0,\ \forall\, t \in [T_1, S_n], \qquad \|v_n(t)\|_{H^s} \leq C_s e^{-\theta_s t}, \tag{2.67}$$

in the case where $s_0 = \infty$.

If g is $\mathscr{C}^\infty$ on $\mathbb{C}$ as an $\mathbb{R}$-differentiable function, it is in particular of class $\mathscr{C}^{\lfloor \frac{d}{2} \rfloor + 2}$, so that we can apply the previous result: there exists $T_1 \geq T_0$ such that for all $n \in \mathbb{N}$,

$$\forall\, t \in [T_1, S_n], \qquad \|v_n(t)\|_{H^{\lfloor \frac{d}{2} \rfloor + 1}} \leq C_{\lfloor \frac{d}{2} \rfloor + 1} e^{-\theta_{\lfloor \frac{d}{2} \rfloor + 1} t}. \tag{2.68}$$

Let $s \geq \left\lfloor \frac{d}{2} \right\rfloor + 2$ and assume that for all $s' \in \{\lfloor \frac{d}{2} \rfloor + 2, \dots, s\}$,

$$\forall\, t \in [T_1, S_n], \qquad \|v_n(t)\|_{H^{s'-1}} \leq C_{s'-1} e^{-\theta_{s'-1} t}.$$

Then define

$$S_{n,s}^* := \inf\{t \geq T_1 \mid \forall\, \tau \in [t, S_n],\ \|v_n(\tau)\|_{H^s} \leq A_s\},$$

for some constant $A_s > \max\{2K\mu_s, 1\}$ to be determined. We show exactly as before (that is considering the functionals $G_{n,s}$) the existence of $\tilde{C}_s > 0$ independent of n, t, and A_s such that

$$\forall\, t \in [S_{n,s}^*, S_n], \qquad \|v_n(t)\|_{H^s}^2 \leq \tilde{C}_s^2 A_s e^{-2\theta_s t}, \tag{2.69}$$

or also

$$\forall\, t \in [S_{n,s}^*, S_n], \qquad \|v_n(t)\|_{H^s} \leq \tilde{C}_s A_s^{\frac{1}{2}} e^{-\theta_s t}. \tag{2.70}$$

Indeed, the constant A in (2.41) does not depend on s and $A_s \geq 1$ so that we have for example as in (2.44) and then (2.46):

$$|I_{\alpha,1}| \leq C A_s e^{-\frac{4\theta}{d+1} t},$$

with C independent of A_s.
Choosing $A_s > \tilde{C}_s^2 e^{-2\theta_s T_1}$ and arguing as in (2.65), we conclude that $S_{n,s}^* = T_1$. Hence, T_1 is uniform with respect to s.

Step 2.3: Looking for optimal exponential decay rates in the uniform H^s-estimates

The next result uses and improves that of Proposition 2.8.

Proposition 2.11. *For all $s \in \{1, \ldots, s_0\}$, there exists $\tilde{A}_s \geq 0$ such that for all $t \in [T_1, S_n]$,*

$$\|v_n(t)\|_{H^s} \leq \tilde{A}_s e^{-\frac{2\theta}{s+1}t}. \tag{2.71}$$

Proof. Let $s' \in \{1, \ldots, s\}$. By (2.23), (2.26), and the following interpolation inequality

$$\|v_n(t)\|_{H^{s'}} \leq \|v_n(t)\|_{L^2}^{\gamma} \|v_n(t)\|_{H^s}^{1-\gamma},$$

with $\gamma = \frac{s-s'}{s}$, we have for all $t \in [T_1, S_n]$,

$$\|v_n(t)\|_{H^{s'}} \leq C e^{-2\theta \frac{s-s'}{s} t}.$$

Now, set

$$T_n^* := \inf\{t \geq T_1 \mid \forall\, \tau \in [t, S_n],\ \|v_n(\tau)\|_{H^s} \leq \tilde{A}_s e^{-\mu\tau}\},$$

for some $\mu \in (0, 2\theta)$ and for some $\tilde{A}_s \geq 1$ to be determined later.
Let t belong to $[T_n^*, S_n]$. Then by the proof set up before,

$$\|v_n(t)\|_{H^s}^2 \leq C \left(\|v_n(t)\|_{H^{s-1}} + e^{-2\theta t} \right).$$

In addition, we obtain once again by interpolation

$$\|v_n(t)\|_{H^{s-1}} \leq \|v_n(t)\|_{L^2}^{\frac{1}{s}} \|v_n(t)\|_{H^s}^{\frac{s-1}{s}} \leq C \tilde{A}_s^{1-\frac{1}{s}} e^{-\frac{2\theta}{s}t} e^{-\mu \frac{s-1}{s} t}.$$

Since $\mu \leq 2\theta$, we have $\frac{2\theta + \mu(s-1)}{s} \leq 2\theta$, and so there exists $\tilde{C} \geq 0$ (independent of $\tilde{A}_s$) such that

$$\|v_n(t)\|_{H^s}^2 \leq \tilde{C} \tilde{A}_s^{1-\frac{1}{s}} e^{-\frac{2\theta}{s}t} e^{-\mu \frac{s-1}{s} t}. \tag{2.72}$$

Now, choose

$$\mu := \frac{2\theta}{s+1} \quad \text{and} \quad \tilde{A}_s > \tilde{C}^{\frac{s}{s+1}}.$$

By a similar argument as that set up to prove Proposition 2.8, we see that $T_n^* = T_1$. Indeed, if we had $T_n^* > T_1$, then by the definition of T_n^* and by continuity of v_n in T_n^*, we would obtain

$$\tilde{A}_s^2 e^{-2\mu T_n^*} = \|v_n(T_n^*)\|_{H^s}^2 \leq \tilde{C} \tilde{A}_s^{1-\frac{1}{s}} e^{-\frac{2\theta}{s} T_n^*} e^{-\mu \frac{s-1}{s} T_n^*},$$

thus, by the choice of μ,

$$\tilde{A}_s^2 \leq \tilde{C} \tilde{A}_s^{1-\frac{1}{s}},$$

which is a contradiction. Consequently, estimate (2.71) does indeed hold. □

2.2.3 Step 3: Conclusion of the proof of Theorem 2.2′

We construct now the multi-soliton u using the same arguments as those of Martel [63, paragraph 2, Step 2] and Martel and Merle [71, Paragraph 2]. The crucial point is the following lemma, obtained by a compactness argument.

Lemma 2.12. *There exist $\varphi \in H^{s_0}(\mathbb{R}^d)$ and a subsequence $(u_{n_k}(T_1))_k$ of $(u_n(T_1))_n$ such that*

$$\|u_{n_k}(T_1) - \varphi\|_{H^{s_0}} \underset{k\to+\infty}{\longrightarrow} 0.$$

Note that the main ingredients to show this lemma are:

- the uniform H^{s_0}-estimate obtained in Step 2.
- the following L^2-compactness assertion: for all $\epsilon > 0$, there exists $\mathcal{K}$ a compact subset of $\mathbb{R}^d$ such that
$$\forall\, n \in \mathbb{N}, \qquad \int_{\mathcal{K}^\complement} |u_n(T_1, x)|^2\, dx \le \epsilon.$$

Then by local well-posedness of (NLS) in $H^{s_0}(\mathbb{R}^d)$ with continuous dependence on compact sets of time [21, Theorem 1.6], the solution u of (NLS) such that $u(T_1) = \varphi$ is defined in $H^{s_0}(\mathbb{R}^d)$ and for all $t \ge T_1$, $\left\|u_{n_k}(t) - \varphi(t)\right\|_{H^{s_0}} \to 0$ as $k \to +\infty$. Thus φ turns out to be the desired multi-soliton. Besides, the quantities $\|u(t) - R(t)\|_{H^s}$ decrease exponentially; this result is obtained by passing to the limit as k tends to $+\infty$ in the H^s-uniform estimates given by Proposition 2.11, that is for all $s = 1, \dots, s_0$, for k large enough:

$$\|u_{n_k}(t) - R(t)\|_{H^s} \le C_s e^{-\frac{2\theta}{s+1}t}.$$

This yields precisely (2.9).

Note that in the case where g is $\mathscr{C}^\infty$ (for example when we consider the pure power non-linearity with p an odd integer), we obtain (2.10) as a consequence of (2.9), by interpolating the corresponding H^s-estimates, and by the independence of T_1 with respect to s proved in Step 2.2.

2.3 Conditional uniqueness for multi-solitons of (NLS)

In this section, we prove the uniqueness result stated in Theorem 2.3′, that is for $d \le 3$. The strategy developed here would also work to prove Proposition 2.6 under the corresponding stronger assumptions.

Our uniqueness result holds due to the coercivity properties of the linearized operators around ground states, namely assumption (H3) when f is not the L^2-critical non-linearity and (2.142) in Proposition 2.25 in the L^2-critical pure power non-linearity case. The proof follows essentially the same lines in these two cases; the differences are only rooted in the use of the appropriate coercivity result.

We first develop the proof in the stable case, assuming $\tilde{f} : z \mapsto f(|z|^2)$ of class $\mathscr{C}^2$ satisfying (2.19). This covers in particular the L^2-subcritical assumption with $3 \le p < 1 + \frac{4}{d}$ and $d = 1$ in Theorem 2.3. In subsection 2.3.4, we explain how to modify the calculations in order to perform the proof in the L^2-critical case, that is assuming $p = 1 + \frac{4}{d}$ and $f : r \mapsto r^{\frac{p-1}{2}}$, which will extend the uniqueness result as stated in Theorems 2.3 and 2.3′.

Let us denote φ the multi-soliton of (NLS) constructed in Theorem 2.1 for $d = 1$ and in Theorem 2.2′ for $d \geq 2$ (which is possible to consider by hypothesis). Set $\gamma := \frac{2\theta}{3}$, where θ is defined in Theorem 2.1 and let $T_1 > 0$ such that φ belongs to $\mathscr{C}([T_1, +\infty), H^1(\mathbb{R}))$ and

$$\forall\, t \geq T_1, \qquad \|\varphi(t) - R(t)\|_{H^1} \leq Ce^{-\gamma t} \tag{2.73}$$

for $d = 1$, and such that φ belongs to $\mathscr{C}([T_1, +\infty), H^{s_0}(\mathbb{R}^d))$ and

$$\forall\, t \geq T_1, \qquad \|\varphi(t) - R(t)\|_{H^{s_0}} \leq Ce^{-\gamma t} \tag{2.74}$$

for $d \geq 2$ (where $s_0 = \lfloor \frac{d}{2} \rfloor + 1$.) In particular, due to the Sobolev embedding $H^{\lfloor \frac{d}{2} \rfloor + 1}(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d)$ for $(d \geq 1)$, we emphasize that, for all $d \geq 1$, $\varphi \in \mathscr{C}([T_1, +\infty[, L^\infty(\mathbb{R}^d))$ and

$$\forall\, t \geq T_1, \qquad \|\varphi(t) - R(t)\|_{L^\infty} \leq Ce^{-\gamma t}. \tag{2.75}$$

Now, let us take u in the class of multi-solitons satisfying (2.11) and define $z := u - \varphi$ the difference of the two multi-solitons so that

$$\partial_t z = i\left(\Delta z + f\left(|z+\varphi|^2\right) z + \left(f\left(|z+\varphi|^2\right) - f\left(|\varphi|^2\right)\right)\varphi\right), \tag{2.76}$$

and

$$\|z(t)\|_{H^1} = \mathrm{O}\left(\frac{1}{t^N}\right), \qquad \text{as } t \longrightarrow +\infty, \tag{2.77}$$

for some integer $N \geq 1$ to be determined later.

We will show that $z = 0$. The idea is to practice some kind of modulation of the variable z in order to ensure some orthogonality relations, needed to make use of the coercivity properties mentioned before. In other words, we obtain a new function (denoted by $\tilde{z}$) which seems to be adapted to the proof; this is the aim of subsection 2.3.1. Then, the control of the modulation parameters allows us to obtain an estimate of $\|z(t)\|_{H^1}$ in terms of $\|\tilde{z}(t)\|_{H^1}$; this combined with the estimate of the derivative of some kind of Weinstein functional $F_{\tilde{z}}$ (that we introduce in paragraph 2.3.2) enables us finally to see that $\tilde{z} = 0$.

2.3.1 Change of function to ensure a coercivity property in the stable case

Introduction of a new variable

We introduce a new function $\tilde{z}$ on $[T, +\infty) \times \mathbb{R}^d$ for T sufficiently large by

$$\forall\, (t,x) \in [T, +\infty) \times \mathbb{R}^d, \quad \tilde{z}(t,x) := z(t,x) + \sum_{k=1}^{K} \left(ia_k(t) R_k(t,x) + b_k(t) \cdot \nabla R_k(t,x)\right), \tag{2.78}$$

where $a_k(t) \in \mathbb{R}$ and $b_k(t) \in \mathbb{R}^d$ are chosen so that

$$\forall\, k \in \{1, \ldots, K\}, \quad \forall\, i \in \{1, \ldots, d\}, \qquad \begin{cases} \operatorname{Im} \displaystyle\int_{\mathbb{R}^d} \tilde{z}\overline{R_k}\, dx = 0 \\ \operatorname{Re} \displaystyle\int_{\mathbb{R}^d} \tilde{z}\partial_{x_i}\overline{R_k}\, dx = 0. \end{cases} \tag{2.79}$$

Existence of $a_k(t)$ and $b_k(t)$ for t large enough follows from:

Lemma 2.13. *For t large enough, and for all $k = 1, \dots, K$, $a_k(t)$ and $b_k(t)$ are uniquely determined. Moreover, $t \mapsto a_k(t)$ and $t \mapsto b_k(t)$ are differentiable in the sense of distributions and*

$$|a_k(t)|, |b_k(t)| \leq C\|z(t)\|_{L^2}, \tag{2.80}$$

$$|a_k'(t)|, |b_k'(t)| \leq C\|z(t)\|_{H^1}. \tag{2.81}$$

Proof of Lemma 2.13. Let us introduce the symmetric block matrix

$$M(t) := \begin{bmatrix} A_{0,0}(t) & B_{1,1}(t) & \cdots & B_{1,d}(t) \\ {}^tB_{1,1}(t) & A_{1,1}(t) & \cdots & A_{1,d}(t) \\ \vdots & \vdots & \ddots & \vdots \\ {}^tB_{1,d}(t) & A_{d,1}(t) & \cdots & A_{d,d}(t) \end{bmatrix},$$

where $A_{i,j}(t)$ and $B_{1,j}(t)$ are $K \times K$-matrices with real entries defined by

$$\begin{aligned} A_{0,0} &= \left[\operatorname{Re}\int_{\mathbb{R}^d} R_k\overline{R_l}\,dx\right]_{(k,l)}, \\ \forall\,(i,j) \in \{1,\dots,d\}^2, \quad A_{i,j} &= \left[\operatorname{Re}\int_{\mathbb{R}^d} \partial_{x_i}R_k\partial_{x_j}\overline{R_l}\,dx\right]_{(k,l)}, \\ \forall\, j \in \{1,\dots,d\}, \quad B_{1,j} &= \left[\operatorname{Im}\int_{\mathbb{R}^d} \partial_{x_j}R_k\overline{R_l}\,dx\right]_{(k,l)}. \end{aligned}$$

Set also

$$x(t) ={}^t \left[a_1, \dots, a_K, b_{1,1}, \dots, b_{K,1}, \dots, b_{1,d}, \dots, b_{K,d}\right]$$

and

$$y(t) = -{}^t\left[y_0, y_1, \dots, y_d\right],$$

where

$$y_0 = \left[\operatorname{Im}\int_{\mathbb{R}^d} z\overline{R_1}\,dx, \dots, \operatorname{Im}\int_{\mathbb{R}^d} z\overline{R_K}\,dx\right]$$

and for all $i = 1, \dots, d$,

$$y_i = \left[\operatorname{Re}\int_{\mathbb{R}^d} z\partial_{x_i}\overline{R_1}\,dx, \dots, \operatorname{Re}\int_{\mathbb{R}^d} z\partial_{x_i}\overline{R_K}\,dx\right].$$

Then relations (2.79) rewrite clearly

$$M(t)x(t) = y(t).$$

Consequently, we have to show that $\det M(t) \neq 0$ for t large enough to ensure existence and uniqueness of $a_k(t)$ and $b_k(t)$ for those values of t. To do this, observe that

$$\operatorname{Re}\int_{\mathbb{R}^d} R_k(t)\overline{R_l}(t)\,dx = \begin{cases} \int_{\mathbb{R}^d} Q_{\omega_k}^2\,dx & \text{if } k = l \\ \mathrm{O}\left(e^{-\gamma t}\right) & \text{if } k \neq l, \end{cases}$$

$$\operatorname{Re}\int_{\mathbb{R}^d} \partial_{x_i} R_k(t) \partial_{x_j} \overline{R_l}(t)\, dx = \begin{cases} \int_{\mathbb{R}^d} \left\{ \partial_{x_i} Q_{\omega_k} \partial_{x_j} Q_{\omega_k} + \frac{v_{i,k} v_{j,k}}{4} Q_{\omega_k}^2 \right\} dx & \text{if } k = l \\ \mathrm{O}\left(e^{-\gamma t}\right) & \text{if } k \neq l, \end{cases}$$

$$\operatorname{Im}\int_{\mathbb{R}^d} \partial_{x_i} R_k(t) \overline{R_l}(t)\, dx = \begin{cases} \frac{v_{i,k}}{2} \int_{\mathbb{R}^d} Q_{\omega_k}^2\, dx & \text{if } k = l \\ \mathrm{O}\left(e^{-\gamma t}\right) & \text{if } k \neq l. \end{cases}$$

Let us now compute $\det(M(t))$. For all $k = 1, \ldots, K$, let L_k denote the k-th line of the block matrix

$$\left[A_{0,0}(t)\ B_{1,1}(t)\ \cdots\ B_{1,d}(t)\right].$$

For all $i = 1, \ldots, d$, and for all $k = 1, \ldots, K$, replacing the k-th line $L_{i,k}$ of the block matrix $\left[{}^t B_{1,i}(t)\ A_{i,1}(t)\ \cdots\ A_{i,d}(t)\right]$ by $L_{i,k} - \frac{v_{i,k}}{2} L_k$, we obtain $\det(M(t)) = \det(N(t))$ where

$$N(t) := \begin{bmatrix} A_{0,0}(t) & B_{1,1}(t) & \cdots & B_{1,d}(t) \\ C_1(t) & D_{1,1}(t) & \cdots & D_{1,d}(t) \\ \vdots & \vdots & \ddots & \vdots \\ C_d(t) & D_{d,1}(t) & \cdots & D_{d,d}(t) \end{bmatrix}$$

and $C_i(t)$ has entries zero on the diagonal and $O(e^{-\gamma t})$ everywhere else and $D_{i,j}$ has entries

$$\int_{\mathbb{R}^d} \partial_{x_i} Q_{\omega_k} \partial_{x_j} Q_{\omega_k}\, dx$$

on the diagonal and $O(e^{-\gamma t})$ everywhere else.
Thus

$$\det(M(t)) = \left(\prod_{k=1}^K \int_{\mathbb{R}^d} Q_{\omega_k}^2(x)\, dx \right) \det(D(t)) + \mathrm{O}(e^{-\gamma t}), \tag{2.82}$$

where $D(t)$ is the sub-matrix of $N(t)$ with block matrices $D_{i,j}(t)$.
We observe that $D(t)$ admits a limit as $t \longrightarrow +\infty$ which we denote by $D(\infty)$ and which corresponds to the block matrix

$$\begin{bmatrix} D_{1,1}(\infty) & \cdots & D_{1,d}(\infty) \\ \vdots & \ddots & \vdots \\ D_{d,1}(\infty) & \cdots & D_{d,d}(\infty) \end{bmatrix}$$

where $D_{i,j}(\infty)$ is a diagonal matrix with entries $\int_{\mathbb{R}^d} \partial_{x_i} Q_{\omega_k} \partial_{x_j} Q_{\omega_k}\, dx$. Due to the continuity of the determinant, $\det(D(t)) \longrightarrow \det D(\infty)$ as $t \longrightarrow +\infty$. Thus,

$$\det M(t) \longrightarrow \left(\prod_{k=1}^K \int_{\mathbb{R}^d} Q_{\omega_k}^2(x)\, dx \right) \det(D(\infty)), \qquad \text{as } t \longrightarrow +\infty. \tag{2.83}$$

Moreover for all $Y = (y_{i,k}) \in \mathbb{R}^{dK}$ different from 0, we have

$${}^t Y D(\infty) Y = \sum_{i,j=1}^d \sum_{k=1}^K \left(\int_{\mathbb{R}^d} \partial_{x_i} Q_{\omega_k} \partial_{x_j} Q_{\omega_k}\, dx \right) y_{i,k} y_{j,k}$$

$$= \sum_{k=1}^{K} \int_{\mathbb{R}^d} \left(\sum_{i=1}^{d} y_{i,k} \partial_{x_i} Q_{\omega_k} \right)^2 \, dx$$

which is a positive quantity for large values of t since for all k, the d functions $\partial_{x_i} Q_{\omega_k}$, $i = 1, \ldots, d$ are linearly independent (this can be seen using that Q_{ω_k} is radial but it is in fact also related to a more general result corresponding to Proposition 2.30 in Appendix).

Hence, $\det(D\infty)) > 0$ and also $\det(M(t)) > 0$ for large values of t by (2.83). In particular, $M(t)$ is invertible for large values of t. Applying Cramer's formula, we obtain an explicit expression of $a_k(t)$ and $b_k(t)$ in terms of $z(t)$, from which we derive the content of Lemma 2.13. Let us justify it.

The entries of $M(t)$ are bounded functions of t so that the transpose of the comatrix of $M(t)$ is bounded too (with respect to t). In addition, we have proved the existence of $c > 0$ such that for t large, $\det M(t) > c$. Hence, there exists $c_0 > 0$ such that for all t sufficiently large,

$$|x(t)| \leq c_0 |y(t)|.$$

This immediately implies (2.80).

We moreover observe that the entries of $M(t)$ are $\mathscr{C}^1$ functions of t (by (2.5) and Lebesgue's dominated convergence theorem); in particular $t \mapsto \det M(t)$ is $\mathscr{C}^1$. Then, the differentiability of $a(t)$ and $b(t)$ and estimate (2.81) follow from the differentiability in the sense of distributions (and the expressions of the differentials) of $t \mapsto \operatorname{Im} \int_{\mathbb{R}^d} z(t)\overline{R_k}(t) \, dx$ and $t \mapsto \operatorname{Re} \int_{\mathbb{R}^d} z(t)\partial_{x_i}(t)\overline{R_k} \, dx$ for $i \in \{1, \ldots, d\}$ and $k \in \{1, \ldots, K\}$. Let us explain how to show the differentiability of $t \mapsto \int_{\mathbb{R}^d} z(t,x)\overline{R_k}(t,x) \, dx$. This is essentially due to a density argument and the local well-posedness of (2.76) with continuous dependence on compact sets of time (as for (NLS)). Let us consider a $\mathscr{C}^1$ function ϕ defined for large values of t and with compact support, say included in $[t_0, t_1]$. Since $z(t_0) \in H^1(\mathbb{R}^d)$, there exists $(z_n(t_0)) \in \mathscr{C}_c^\infty(\mathbb{R}^d)$ converging to $z(t_0)$ in the sense of the H^1-norm. The solution z_n of (2.76) with initial data $z_n(t_0)$ in time t_0 is defined on $[t_0, t_1])$ for n large, belongs to $\mathscr{C}\left([t_0, t_1], \mathscr{S}(\mathbb{R}^d)\right)$, and satisfies

$$\sup_{t \in [t_0, t_1]} \|z_n(t) - z(t)\|_{H^1} \to 0, \qquad \text{as } n \to +\infty. \tag{2.84}$$

Now, by Fubini theorem and the differentiability of $t \mapsto z_n(t,x)R_k(t,x)$ for all $x \in \mathbb{R}^d$, we obtain

$$\begin{aligned}
&\int_{t_0}^{t_1} \left(\int_{\mathbb{R}^d} z_n(t,x)R_k(t,x) \, dx \right) \phi'(t) \, dt \\
&= \int_{\mathbb{R}^d} \left(\int_{t_0}^{t_1} z_n(t,x)R_k(t,x)\phi'(t) \, dt \right) dx \\
&= -i \int_{\mathbb{R}^d} \left(\int_{t_0}^{t_1} \left((\Delta z_n + g(z_n + \phi) - g(\phi))R_k + z_n \partial_t R_k \right)(t,x)\phi(t) \, dt \right) dx \\
&= -i \int_{t_0}^{t_1} \phi(t) \left(\int_{\mathbb{R}^d} \left((\Delta z_n + g(z_n + \phi) - g(\phi))R_k + z_n \partial_t R_k \right)(t,x) \, dx \right) dt \\
&= -i \int_{t_0}^{t_1} \phi(t) \left(\int_{\mathbb{R}^d} \left(\nabla z_n \cdot R_k + (g(z_n + \phi) - g(\phi))R_k + z_n \partial_t R_k \right)(t,x) \, dx \right) dt.
\end{aligned}$$

(We recall that $g(z) = zf(|z|^2)$ for $z \in \mathbb{C}$.) Passing to the limit as $n \to +\infty$ by using (2.84) leads to

$$\int_{t_0}^{t_1} \left(\int_{\mathbb{R}^d} z(t,x) R_k(t,x)\ dx \right) \phi'(t)\ dt$$
$$= -i \int_{t_0}^{t_1} \phi(t) \left(\int_{\mathbb{R}^d} \left(\nabla z \cdot R_k + g(z+\phi) - g(\phi) R_k + z \partial_t R_k \right)(t,x)\ dx \right)\ dt. \quad (2.85)$$

Thus $t \mapsto \int_{\mathbb{R}^d} z\overline{R_k}(t,x)\ dx$ is differentiable in the sense of distributions; its differential is

$$t \mapsto \int_{\mathbb{R}^d} \left(\nabla z \cdot R_k + (g(z+\phi) - g(\phi)) R_k + z \partial_t R_k \right)(t,x)\ dx$$

and is thus bounded by $\|z(t)\|_{H^1}$.
This finishes proving the lemma. □

Even if it means taking a larger T_1, we can suppose that the preceding lemma holds on $[T_1, +\infty)$. Then it results also immediately that

$$\forall\ t \geq T_1, \qquad \|z(t)\|_{H^1} \leq \|\tilde{z}(t)\|_{H^1} + C \sum_{k=1}^{K} \left(|a_k(t)| + |b_k(t)| \right). \quad (2.86)$$

and

$$\forall\ t \geq T_1, \qquad \|\tilde{z}(t)\|_{H^1} \leq C \|z(t)\|_{H^1}. \quad (2.87)$$

The statement of a coercivity property in terms of the new variable

In this paragraph, we come to some crucial inequality, on which the proof is essentially based. First of all, let us define some notations, and particularly well-chosen cut-off functions.

By a classical argument given in [71, Claim 1], we can assume (without loss of generality) that

$$v_{1,1} < v_{2,1} < \cdots < v_{K,1}. \quad (2.88)$$

Now let $A_0 \in \left] 0, \frac{1}{2} \min_{k \in \{2,\dots,K\}} \{ v_{k,1} - v_{k-1,1} \} \right[$ and define

$$\psi : \mathbb{R} \to \mathbb{R}$$
$$x \mapsto \begin{cases} 1 & \text{if } x < -A_0 \\ \left(\int_{-A_0}^{A_0} e^{-\frac{A_0^2}{A_0^2 - y^2}}\ dy \right)^{-1} \int_x^{A_0} e^{-\frac{A_0^2}{A_0^2 - y^2}}\ dy & \text{if } -A_0 \leq x \leq A_0 \\ 0 & \text{if } x > A_0, \end{cases}$$

which is obviously a smooth bounded non-increasing function.
For all $k \in \{1, \dots, K-1\}$, let $\sigma_k := \frac{1}{2} \left(v_{k,1} + v_{k+1,1} \right)$ and $\xi_k := \frac{1}{2} \left(x^0_{k,1} + x^0_{k+1,1} \right)$. Then define on $\mathbb{R} \times \mathbb{R}^d$

$$\psi_0 : (t,x) \mapsto 0$$
$$\psi_k : (t,x) \mapsto \psi\left(\frac{x_1 - \xi_k - \sigma_k t}{t} \right) \quad \text{for } k \in \{1, \dots, K-1\}$$
$$\psi_K : (t,x) \mapsto 1$$

and also K functions on $\mathbb{R} \times \mathbb{R}^d$ by

$$\forall\, k \in \{1, \ldots, K\}, \quad \phi_k := \psi_k - \psi_{k-1}.$$

We can check that, for large values of t, $\phi_k(t, \cdot)$ has a smooth profile localized at the "neighborhood" of the k-th solitary wave; more precisely we have

$$\phi_1(t,x) = \begin{cases} 1 & \text{if } x_1 < \xi_1 + (\sigma_1 - A_0)t \\ 0 & \text{if } x_1 > \xi_1 + (\sigma_1 + A_0)t, \end{cases}$$

for all $k = 2, \ldots, K-1$:

$$\phi_k(t,x) = \begin{cases} 1 & \text{if } \xi_{k-1} + (\sigma_{k-1} + A_0)t < x_1 < \xi_k + (\sigma_k - A_0)t \\ 0 & \text{if } x_1 < \xi_{k-1} + (\sigma_{k-1} - A_0)t \text{ or } x_1 > \xi_k + (\sigma_k + A_0)t, \end{cases}$$

and

$$\phi_K(t,x) = \begin{cases} 1 & \text{if } \xi_{K-1} + (\sigma_{K-1} + A_0)t < x_1 \\ 0 & \text{if } x_1 < \xi_{K-1} + (\sigma_{K-1} - A_0)t. \end{cases}$$

Besides, for large values of t, the following inequalities hold owing to the decay properties of R_k and the support properties of ϕ_j and its derivatives.

Lemma 2.14. *Even if it means reducing $\gamma > 0$ so that*

$$\gamma < \min\left\{ \frac{\sqrt{\omega_k}}{4} \left(\frac{v_{j,1} - v_{j-1,1}}{2} - A_0 \right),\ k = 1, \ldots, K,\ j = 2, \ldots, K \right\},$$

we have:

$$\forall\, j \neq k, \qquad \big(|R_k(t,x)| + |\partial_{x_1} R_k(t,x)|\big)|\phi_j(t,x)| \leq C e^{-\gamma t} e^{-\frac{\sqrt{\omega_k}}{4}|x - v_k t|} \tag{2.89}$$

$$\forall\, j, \qquad \big(|R_j(t,x)| + |\partial_{x_1} R_j(t,x)| + |\partial_t R_j(t,x)|\big)|\phi_j(t,x) - 1| \leq C e^{-\gamma t} e^{-\frac{\sqrt{\omega_j}}{4}|x - v_j t|} \tag{2.90}$$

$$\forall\, j, \qquad |\partial_{x_1} \phi_j(t,x)| + |\partial_{x_1}^3 \phi_j(t,x)| + |\partial_t \phi_j(t,x)| \leq \frac{C}{t}. \tag{2.91}$$

$$\forall\, j,k, \qquad \big(|R_k(t,x)| + |\partial_{x_1} R_k(t,x)|\big) \left|\partial_{x_1} \phi_j(t,x)\right| \leq C e^{-\gamma t} e^{-\frac{\sqrt{\omega_k}}{4}|x - v_k t|}. \tag{2.92}$$

Proof. The proof, postponed in Appendix, is similar to that of Combet [12, Proof of Lemma 3.9, Appendix A]. □

Let us introduce the following Weinstein energy functional which is inspired from Martel, Merle and Tsai [81] for dimensions 1 to 3:

$$H(t) := \sum_{k=1}^{K} \int_{\mathbb{R}^d} \Big\{ |\nabla \tilde{z}|^2 - \Big(f\left(|R_k|^2\right) |\tilde{z}|^2 + 2\mathrm{Re}(\overline{R_k}\tilde{z})^2 f'\left(|R_k|^2\right) \Big) + \left(\omega_k + \frac{|v_k|^2}{4} \right) |\tilde{z}|^2 - v_k \cdot \mathrm{Im}\left(\nabla \tilde{z} \overline{\tilde{z}} \right) \Big\} \phi_k(t,x)\, dx. \tag{2.93}$$

One of the main features concerning H is the following coercivity property, which turns out to be a key ingredient in our matter.

Proposition 2.15. *There exists* $C > 0$ *such that*

$$\forall\, t \geq T_1, \quad C\|\tilde{z}(t)\|_{H^1}^2 - \frac{1}{C}\sum_{k=1}^{K}\left(\operatorname{Re}\int_{\mathbb{R}^d} \tilde{z}(t)\overline{R_k}(t)\, dx\right)^2 \leq H(t). \tag{2.94}$$

Proof. This result follows from our assumption (H3), from (2.79), and an immediate adaptation to all dimensions of the proof given for the one-dimensional case in [81, appendix B] which consists in localizing in some sense each version of (H3) for all $k = 1, \dots, K$. □

2.3.2 Proof of some needed estimates

This subsection, which is probably the most technical one, precises the tools and estimates which will allow us to make use of Proposition 2.15 and actually to conclude the proof of uniqueness in subsection 2.3.3. It consists in giving some controls of $H(t)$, of the scalar products

$$\operatorname{Re}\int_{\mathbb{R}^d} \tilde{z}(t)\overline{R_k}(t)\, dx,$$

and also of the modulation parameters $a_k(t)$ and $b_k(t)$.

Control of H

We typically improve the a priori control of H by $\mathrm{O}\left(\|\tilde{z}\|_{H^1}^2\right)$ by differentiation of the functional. Actually, for the sake of simplification, we will compute the derivative of the following related functional $\tilde{H} : [T_1, +\infty) \to \mathbb{R}$ defined by $\forall\, t \geq T_1$,

$$\tilde{H}(t) = \sum_{k=1}^{K}\int_{\mathbb{R}^d}\left\{|\nabla\tilde{z}|^2 - \left(F(|\tilde{z}+\varphi|^2) - F(|\varphi|^2) - 2\operatorname{Re}(\tilde{z}\overline{\varphi})f(|\varphi|^2)\right)\right.$$
$$\left. + \left(\omega_k + \frac{|v_k|^2}{4}\right)|\tilde{z}|^2 - v_k\cdot\operatorname{Im}\left(\nabla\tilde{z}\bar{\tilde{z}}\right)\right\}\phi_k(t,x)\, dx. \tag{2.95}$$

(Recall that $F(r) = \int_0^r f(\rho)\, d\rho$ (2.12).)

The next proposition, which compares H and $\tilde{H}$, justifies that it suffices to control $\tilde{H}$ in order to obtain a similar estimate for H.

Proposition 2.16. *We have*

$$H(t) = \tilde{H}(t) + \mathrm{O}\left(\|\tilde{z}(t)\|_{H^1}^3 + e^{-\gamma t}\|\tilde{z}(t)\|_{H^1}^2\right). \tag{2.96}$$

Proof. Let us first observe that $\tilde{F} : z \mapsto F(|z|^2)$ is $\mathscr{C}^3$ on $\mathbb{C}$. Indeed, since $\tilde{f}$ is $\mathscr{C}^2$ on $\mathbb{C}$, the function f is $\mathscr{C}^2$ on $(0, +\infty)$ and thus, $\tilde{F}$ is $\mathscr{C}^3$ on $\mathbb{C}\setminus\{0\}$. Moreover, for all $z = (\operatorname{Re}(z), \operatorname{Im}(z)) = (x, y) \in \mathbb{C}\setminus\{0\}$, we obtain by differentiation of $\tilde{f}$ and $\tilde{F}$:

$$\partial_x\tilde{F}(z) = 2xf(|z|^2) = 2x\tilde{f}(z)$$
$$\partial_y\tilde{F}(z) = 2yf(|z|^2) = 2y\tilde{f}(z)$$
$$\partial_{xx}\tilde{F}(z) = 2f(|z|^2) + 4x^2f'(|z|^2) = 2\tilde{f}(z) + 2x\partial_x\tilde{f}(z)$$

$$
\begin{aligned}
\partial_{xy}\tilde{F}(z) &= 4xyf'(|z|^2) = 2y\partial_x \tilde{f}(z)\\
\partial_{yy}\tilde{F}(z) &= 2f(|z|^2) + 4y^2 f'(|z|^2) = 2\tilde{f}(z) + 2y\partial_y \tilde{f}(z)\\
\partial_{xxx}\tilde{F}(z) &= 4\partial_x \tilde{f}(z) + 2x\partial_{xx}\tilde{f}(z)\\
\partial_{xxy}\tilde{F}(z) &= 2y\partial_{xx}\tilde{f}(z)\\
\partial_{xyy}\tilde{F}(z) &= 2x\partial_{yy}\tilde{f}(z)\\
\partial_{yyy}\tilde{F}(z) &= 4\partial_y \tilde{f}(z) + 2y\partial_{yy}\tilde{f}(z).
\end{aligned}
$$

Since $\tilde{f}$ is $\mathscr{C}^2$, the partial differentials of $\tilde{F}$ up to order 3 admit limits as $(x, y) \to (0, 0)$ in $\mathbb{R}^2$, from which we deduce that $\tilde{F}$ is $\mathscr{C}^3$.

Then we have the following Taylor expansion: for $t \geq T_1$ and $x \in \mathbb{R}^d$,

$$
\begin{aligned}
&\left(\tilde{F}(\tilde{z}+\varphi) - \tilde{F}(\varphi) - \mathrm{Re}(\tilde{z})\partial_x \tilde{F}(\varphi) - \mathrm{Im}(\tilde{z})\partial_y \tilde{F}(\varphi)\right)(t,x)\\
&\qquad = \frac{1}{2}\left((\mathrm{Re}\tilde{z})^2 \partial_{xx}\tilde{F} + 2\mathrm{Re}\tilde{z}\mathrm{Im}\tilde{z}\partial_{xy}\tilde{F} + (\mathrm{Im}\tilde{z})^2\partial_{yy}\tilde{F}\right)(t,x) + \rho(t,x), \quad (2.97)
\end{aligned}
$$

where

$$
\begin{aligned}
|\rho(t,x)| &\leq \frac{1}{6}|\tilde{z}|^3 \sup_{\omega\in[\varphi(t,x),(\tilde{z}+\varphi)(t,x)]} \|D^3_\omega \tilde{F}\|\\
&\leq C|\tilde{z}|^3 \sup_{\omega\in[\varphi(t,x),(\tilde{z}+\varphi)(t,x)]} \left(\|D_\omega \tilde{f}\| + |\tilde{z}|\|D^2_\omega \tilde{f}\|\right)\\
&\leq C|\tilde{z}|^3\left(1 + \tilde{z}|^{\frac{4}{d}-1}\right)
\end{aligned}
$$

by assumption (2.19). We then note that the preceding Taylor expansion rewrites

$$
F(|\tilde{z}+\varphi|^2) - F(|\varphi|^2) - 2\mathrm{Re}(\tilde{z}\overline{\varphi})f(|\varphi|^2) = |\tilde{z}|^2 f(|\varphi|^2) + 2\mathrm{Re}(\tilde{z}\overline{\varphi})^2 f'(|\varphi|^2) + \mathrm{O}(|\tilde{z}|^3 + |\tilde{z}|^{\frac{4}{d}+2}), \quad (2.98)
$$

uniformly with respect to both variables t and x. Let us underline that, for $d \geq 2$, one can not claim whether $z(t)$ or $\tilde{z}(t)$ belong to $L^\infty(\mathbb{R}^d)$ and even less whether z or $\tilde{z}$ belong to $L^\infty\left([T_1, +\infty), L^\infty(\mathbb{R}^d)\right)$, which prevents us from simplifying $\mathrm{O}\left(|\tilde{z}|^3 + |\tilde{z}|^{\frac{4}{d}+2}\right)$ by $\mathrm{O}\left(|\tilde{z}|^3\right)$.

Moreover, we have noticed that

$$
|\tilde{z}|^2 f(|\varphi|^2) + 2\mathrm{Re}(\tilde{z}\overline{\varphi})^2 f'(|\varphi|^2) = \frac{1}{2} D^2_\varphi \tilde{F}(\tilde{z}, \tilde{z})
$$

so that

$$
\sum_{k=1}^{K} \int_{\mathbb{R}^d} \left\{|\tilde{z}|^2 f\left(|\varphi|^2\right) + 2\mathrm{Re}(\tilde{z}\overline{\varphi})^2 f'\left(|\varphi|^2\right)\right\}\phi_k \, dx = \sum_{k=1}^{K}\int_{\mathbb{R}^d} D^2_\varphi \tilde{F}(\tilde{z},\tilde{z})\phi_k \, dx.
$$

We now observe that for all $k = 1, \dots, K$,

$$
\begin{aligned}
&\left|\int_{\mathbb{R}^d}\left(D^2_\varphi \tilde{F}(\tilde{z},\tilde{z}) - D^2_{R_k}\tilde{F}(\tilde{z},\tilde{z})\right)\phi_k \, dx\right|\\
&\qquad \leq \int_{\mathbb{R}^d} |(\varphi - R_k)\phi_k||\tilde{z}|^2 \sup_{\omega\in[R_k,\varphi]} \|D^3_\omega \tilde{F}\| \, dx
\end{aligned}
$$

$$\leq \int_{\mathbb{R}^d} |(\varphi - R)\phi_k||\tilde{z}|^2 \sup_{\omega\in[R_k,\varphi]} \|D^3_\omega \tilde{F}\| \, dx + \sum_{j\neq k} \int_{\mathbb{R}^d} |R_j\phi_k||\tilde{z}|^2 \sup_{\omega\in[R_k,\varphi]} \|D^3_\omega \tilde{F}\| \, dx.$$

We note that $\sup_{\omega\in[R_k,\varphi]} \|D^3_\omega \tilde{F}\|$ is bounded by a constant C independent of t and x because R_k, φ belong to $L^\infty([T_1,+\infty), L^\infty(\mathbb{R}^d))$ and $\omega \mapsto D^3_\omega \tilde{F}$ is continuous on $\mathbb{C}$. Then (2.74) and (2.89) lead to

$$\sum_{k=1}^K \int_{\mathbb{R}^d} \left\{|\tilde{z}|^2 f\left(|\varphi|^2\right) + 2\mathrm{Re}(\tilde{z}\overline{\varphi})^2 f'\left(|\varphi|^2\right)\right\} \phi_k \, dx$$
$$= \sum_{k=1}^K \int_{\mathbb{R}^d} \left\{|\tilde{z}|^2 f\left(|R_k|^2\right) + 2\mathrm{Re}(\tilde{z}\overline{R_k})^2 f'\left(|R_k|^2\right)\right\} \phi_k \, dx + \mathrm{O}\left(e^{-\gamma t}\|\tilde{z}\|^2_{H^1}\right). \quad (2.99)$$

We finally obtain (2.96) as a direct consequence of (2.98), (2.99), the Sobolev embeddings $H^1(\mathbb{R}^d) \hookrightarrow L^3(\mathbb{R}^d)$ (indeed available for $d \leq 3$) and $H^1(\mathbb{R}^d) \hookrightarrow L^{\frac{4}{d}+2}(\mathbb{R}^d)$, and the fact that $\frac{4}{d} \geq 1$.

□

Now, we state and prove the crucial

Proposition 2.17. *The derivative of $\tilde{H}$ is given by*

$$\frac{d}{dt}\tilde{H}(t) = Main(t) + \mathrm{O}\big(e^{-\gamma t}\|\tilde{z}(t)\|_{H^1}\|z(t)\|_{H^1} + \|\tilde{z}(t)\|_{H^1}\|z(t)\|^2_{H^1}\big), \qquad as\ t \to +\infty. \quad (2.100)$$

where

$$\begin{aligned} Main(t) := &\sum_{k=1}^K \left(\omega_k + \frac{|v_k|^2}{4}\right)\left(\int_{\mathbb{R}^d} |\tilde{z}|^2 \partial_t\phi_k \, dx + 2\int_{\mathbb{R}^d} \mathrm{Im}(\partial_{x_1}\tilde{z}\overline{\tilde{z}})\partial_{x_1}\phi_k \, dx\right) \\ &- \sum_{k=1}^K v_{k,1}\left(2\int_{\mathbb{R}^d} |\nabla\tilde{z}|^2\partial_{x_1}\phi_k \, dx - \frac{1}{2}\int_{\mathbb{R}^d} |\tilde{z}|^2\partial^3_{x_1}\phi_k \, dx\right) \\ &- \sum_{k=1}^K v_k \cdot \int_{\mathbb{R}^d} \mathrm{Im}\left(\nabla\tilde{z}\overline{\tilde{z}}\right)\partial_t\phi_k \, dx \\ = &\ \mathrm{O}\left(\frac{1}{t}\|\tilde{z}(t)\|^2_{H^1}\right). \end{aligned}$$

Remark 2.7. The bound $\mathrm{O}\left(\frac{1}{t}\|\tilde{z}(t)\|^2_{H^1}\right)$ in above is the one which constrains us to prove uniqueness in the class satisfying (2.11). In order to prove unconditionnal uniqueness, one would need to improve this bound to $\mathrm{O}\left(\alpha(t)\|\tilde{z}(t)\|^2_{H^1}\right)$, where $\alpha(t)$ is integrable in time (in the KdV context, [63] proves it with $\alpha(t) = e^{-\gamma t}$ for some $\gamma > 0$).

Let us begin with some preliminaries (Lemma 2.18 and Lemma 2.19 below), which are needed to obtain Proposition 2.17.

Lemma 2.18. *There exists $C > 0$ such that:*

$$\left|f(|z+\varphi|^2) - f(|\varphi|^2)\right| \leq C\big(|z|^{\frac{4}{d}} + |z|\big)$$
$$\left|f(|z+\varphi|^2) - f(|\varphi|^2) - 2\mathrm{Re}(z\overline{\varphi})f'(|\varphi|^2)\right| \leq C\left(|z|^2 + |z|^{\frac{4}{d}}\right).$$

Proof. By the mean value theorem applied to $\tilde{f}$,

$$\left|\tilde{f}(z+\varphi)-\tilde{f}(\varphi)\right|(t,x) \leq |z(t,x)| \sup_{\omega\in[\varphi(t,x),(z+\varphi)(t,x)]} \|D_\omega \tilde{f}\|$$

By (2.19) and the boundedness of φ with respect to t and x, it results

$$\left|\tilde{f}(z+\varphi)-\tilde{f}(\varphi)\right|(t,x) \leq C\left(|z(t,x)|+|z(t,x)|^{\frac{4}{d}}\right).$$

Similarly, the second estimate stated in Lemma 2.18 is obtained by direct application of Taylor formula for $\tilde{f}$ at order 2 and by using

$$\sup_{\omega\in[\varphi(t,x),(z+\varphi)(t,x)]} \|D^2_\omega \tilde{f}\| \leq C(1+|\tilde{z}|^{\frac{4}{d}-2}).$$

□

Lemma 2.19 (Expression of $\partial_t\tilde{z}$)**.** *We have*

$$\begin{aligned}\partial_t\tilde{z} &= i\Big(\Delta\tilde{z}+f(|z+\varphi|^2)z+\big(f(|z+\varphi|^2)-f(|\varphi|^2)\big)\varphi\Big)\\ &\quad+i\sum_{k=1}^{K}\left\{ia_k f(|R_k|^2)R_k+b_k\cdot\nabla\big(f(|R_k|^2)R_k\big)\right\}+\sum_{k=1}^{K}\left\{ia'_k R_k+b'_k\cdot\nabla R_k\right\}\\ &= i\Big(\Delta\tilde{z}+f(|\varphi|^2)\tilde{z}+\big(f(|\tilde{z}+\varphi|^2)-f(|\varphi|^2)\big)\varphi\Big)+\sum_{k=1}^{K}\left\{ia'_k R_k+b'_k\cdot\nabla R_k\right\}+\epsilon,\end{aligned} \tag{2.101}$$

where ϵ is a function of t and x such that

$$\int_{\mathbb{R}^d}|\epsilon|\big(|\tilde{z}|+|\nabla\tilde{z}|\big)\,dx+\left|\int_{\mathbb{R}^d}\epsilon\Delta\tilde{z}\,dx\right| \leq C\big(e^{-\gamma t}\|z\|_{H^1}\|\tilde{z}\|_{H^1}+\|z\|^2_{H^1}\|\tilde{z}\|_{H^1}\big). \tag{2.102}$$

Proof. The first equality concerning $\partial_t\tilde{z}$ is quite immediate. Let us precise how to obtain the second equality. Decomposing

$$f(|z+\varphi|^2)z = f(|\varphi|^2)z+\big(f(|z+\varphi|^2)-f(|\varphi|^2)\big)z,$$

and using the expression of z in terms of $\tilde{z}$ and the a_k and b_k, $k=1,\dots,K$ given by (2.78), we have that

$$\begin{aligned}&f(|z+\varphi|^2)z+\sum_{k=1}^{K}\left\{ia_k f(|R_k|^2)R_k+b_k\cdot\nabla\left(f(|R_k|^2)R_k\right)\right\}\\ &\qquad= f(|\varphi|^2)\tilde{z}+\sum_{k=1}^{K}\left\{ia_k\big(f(|R_k|^2)-f(|\varphi|^2)\big)R_k+b_k\cdot\nabla R_k\big(f(|R_k|^2)-f(|\varphi|^2)\big)\right\}\\ &\qquad\quad+\sum_{k=1}^{K}b_k\cdot\nabla(f(|R_k|^2))R_k+\big(f(|z+\varphi|^2)-f(|\varphi|^2)\big)z\end{aligned} \tag{2.103}$$

Moreover, using the second estimate obtained in Lemma 2.18,

$$\begin{aligned}
&f(|z+\varphi|^2) - f(|\varphi|^2)\\
&\qquad = f(|\tilde{z}+\varphi|^2) - f(|\varphi|^2) - 2\mathrm{Re}\big((\tilde{z}-z)\overline{\varphi}\big)f'(|\varphi|^2) + \mathrm{O}(|z|^2 + |z|^{\frac{4}{d}})\\
&\qquad = f(|\tilde{z}+\varphi|^2) - f(|\varphi|^2) - 2\sum_{k=1}^{K}\mathrm{Re}\big(b_k\cdot\nabla R_k\overline{R_k}\big)f'(|R_k|^2) + h + \mathrm{O}(|z|^2 + |z|^{\frac{4}{d}})\\
&\qquad = f(|\tilde{z}+\varphi|^2) - f(|\varphi|^2) - \sum_{k=1}^{K} b_k\cdot\nabla\big(f(|R_k|^2) + \tilde{h},
\end{aligned} \tag{2.104}$$

where h and $\tilde{h}$ satisfy the same property (2.102) as ϵ due to (2.74) and (2.89). Lemma 2.19 is now a consequence of (2.103), (2.104), and the fact that $\frac{4}{d}+1\geq 2$ (for $d\leq 3$). □

We are now in a position to prove (2.100).

Proof of Proposition 2.17. The proof decomposes essentially into two parts. We first differentiate successively each term constituting $\tilde{H}$ by means of Lemma 2.19. For this, integrations by parts are sometimes necessary in order not to keep terms carrying second spatial derivatives for z. Then we put together suitable terms in the expression of $\frac{d}{dt}\tilde{H}$ in order to get better estimates than the a priori control by $\mathrm{O}\left(\|\tilde{z}(t)\|^2_{H^1}\right)$. Besides, we put annotations for the different terms we have to work on for ease of reading; terms associated with the same letter A, B, or C are to be gathered.

Step 1: Differentiation of $\tilde{H}$

- Using Lemma 2.19, one computes

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^d}|\nabla\tilde{z}|^2\,dx\\
&\quad = 2\,\mathrm{Re}\int_{\mathbb{R}^d}\nabla\tilde{z}_t\cdot\nabla\overline{\tilde{z}}\,dx\\
&\quad = 2\,\mathrm{Im}\int_{\mathbb{R}^d} f(|\varphi|^2)\tilde{z}\Delta\overline{\tilde{z}}\,dx + 2\,\mathrm{Im}\int_{\mathbb{R}^d}\big(f(|\tilde{z}+\varphi|^2) - f(|\varphi|^2)\big)\varphi\Delta\overline{\tilde{z}}\,dx\\
&\qquad - 2\,\mathrm{Re}\int_{\mathbb{R}^d}\sum_{k=1}^{K}\big\{ia'_kR_k + b'_k\cdot\nabla R_k\big\}\Delta\overline{\tilde{z}}\,dx + \mathrm{O}\big((e^{-\gamma t} + \|z\|_{H^1})\|z\|_{H^1}\|\tilde{z}\|_{H^1}\big).
\end{aligned}$$

Similarly one obtains directly

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^d}\Big\{F(|\tilde{z}+\varphi|^2) - F(|\varphi|^2) - 2\mathrm{Re}(\tilde{z}\overline{\varphi})f(|\varphi|^2)\Big\}\,dx\\
&\quad = 2\,\mathrm{Re}\int_{\mathbb{R}^d}\varphi_t\overline{\varphi}\big(f(|\tilde{z}+\varphi|^2) - f(|\varphi|^2) - 2\mathrm{Re}(\tilde{z}\overline{\varphi})f'(|\varphi|^2)\big)\,dx\\
&\qquad + 2\,\mathrm{Re}\int_{\mathbb{R}^d}\tilde{z}_t\overline{\tilde{z}}f(|\tilde{z}+\varphi|^2)\,dx + 2\,\mathrm{Re}\int_{\mathbb{R}^d}(\varphi_t\overline{\tilde{z}} + \tilde{z}_t\overline{\varphi})\big(f(|\tilde{z}+\varphi|^2) - f(|\varphi|^2)\big)\,dx\\
&\quad = -2\,\mathrm{Im}\int_{\mathbb{R}^d}\Delta\tilde{z}\overline{\tilde{z}}f(|\tilde{z}+\varphi|^2)\,dx
\end{aligned}$$

$$
\begin{aligned}
&- 2\,\mathrm{Im}\int_{\mathbb{R}^d}\Delta\varphi\overline{\varphi}\big(f(|\tilde z+\varphi|^2)-f(|\varphi|^2)-2\mathrm{Re}(\tilde z\overline{\varphi})f'(|\varphi|^2)\big)\,dx\\
&- 2\,\mathrm{Im}\int_{\mathbb{R}^d}\big(f(|\tilde z+\varphi|^2)-f(|\varphi|^2)\big)\varphi\overline{\tilde z}(f(|\tilde z+\varphi|^2)\,dx\\
&- 2\,\mathrm{Im}\int_{\mathbb{R}^d}\Delta\tilde z\overline{\varphi}\big(f(|\tilde z+\varphi|^2)-f(|\varphi|^2)\big)\,dx\\
&- 2\,\mathrm{Im}\int_{\mathbb{R}^d}\big(\Delta\varphi+f(|\varphi|^2)\varphi\big)\overline{\tilde z}\big(f(|\tilde z+\varphi|^2)-f(|\varphi|^2)\big)\,dx\\
&- 2\,\mathrm{Im}\int_{\mathbb{R}^d}f(|\varphi|^2)\tilde z\overline{\varphi}\big(f(|\tilde z+\varphi|^2)-f(|\varphi|^2)\big)\,dx\\
&+ 2\,\mathrm{Re}\int_{\mathbb{R}^d}\sum_{k=1}^{K}\big(ia_k'R_k+b_k'\cdot\nabla R_k\big)\Big(\overline{\tilde z}f(|\tilde z+\varphi|^2)+\overline{\varphi}\big(f(|\tilde z+\varphi|^2)-f(|\varphi|^2)\big)\Big)\,dx\\
&+ \mathrm{O}\big((e^{-\gamma t}+\|z\|_{H^1})\|z\|_{H^1}\|\tilde z\|_{H^1}\big).
\end{aligned}
$$

Thus, we have at this point

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^d}\big\{|\nabla\tilde z|^2-\big(F(|\tilde z+\varphi|^2)-F(|\varphi|^2)-2\mathrm{Re}(\tilde z\overline{\varphi})f(|\varphi|^2)\big)\big\}\,dx\\
&\quad= 2\,\mathrm{Im}\int_{\mathbb{R}^d}\Delta\varphi\overline{\varphi}\big(f(|\tilde z+\varphi|^2)-f(|\varphi|^2)-2\mathrm{Re}(\tilde z\overline{\varphi})f'(|\varphi|^2)\big)\,dx \qquad (B_1)\\
&\qquad+ 2\,\mathrm{Im}\int_{\mathbb{R}^d}\big(\Delta\varphi+f(|\varphi|^2)\varphi\big)\overline{\tilde z}\big(f(|\tilde z+\varphi|^2)-f(|\varphi|^2)\big)\,dx \qquad (A_1)\\
&\qquad- 2\,\mathrm{Re}\int_{\mathbb{R}^d}\sum_{k=1}^{K}\big(ia_k'R_k+b_k'\cdot\nabla R_k\big)\big(\Delta\overline{\tilde z}+\overline{\tilde z}f(|\tilde z+\varphi|^2)\big)\,dx \qquad (C_1)\\
&\qquad- 2\,\mathrm{Re}\sum_{k=1}^{K}\int_{\mathbb{R}^d}b_k'\cdot\nabla R_k\overline{R_k}\big(f(|\tilde z+\varphi|^2)-f(|\varphi|^2)\big)\,dx \qquad (C_2)\\
&\qquad+ \mathrm{O}\left((e^{-\gamma t}+\|z\|_{H^1})\,\|z\|_{H^1}\|\tilde z\|_{H^1}\right).
\end{aligned}
$$

- We differentiate then the next term appearing in the expression of $\tilde H$. For all $k=1,\dots,K$,

$$
\begin{aligned}
&\left(\omega_k+\frac{|v_k|^2}{4}\right)\frac{d}{dt}\int_{\mathbb{R}^d}|\tilde z|^2\phi_k\,dx\\
&\quad=\left(\omega_k+\frac{|v_k|^2}{4}\right)\int_{\mathbb{R}^d}|\tilde z|^2\partial_t\phi_k\,dx+2\left(\omega_k+\frac{|v_k|^2}{4}\right)\int_{\mathbb{R}^d}\mathrm{Im}\left(\partial_{x_1}\tilde z\overline{\tilde z}\right)\partial_{x_1}\phi_k\,dx \qquad (Main_{1,k})\\
&\qquad-2\left(\omega_k+\frac{|v_k|^2}{4}\right)\int_{\mathbb{R}^d}\mathrm{Im}(\varphi\overline{\tilde z})(f(\tilde z+\varphi|^2)-f(|\varphi|^2))\phi_k\,dx \qquad (A_{2,k})\\
&\qquad+\ 2\left(\omega_k+\frac{|v_k|^2}{4}\right)\mathrm{Re}\int_{\mathbb{R}^d}\big(ia_k'R_k+b_k'\cdot\nabla R_k\big)\overline{\tilde z}\phi_k\,dx \qquad (C_{3,k})\\
&\qquad+\mathrm{O}\big((e^{-\gamma t}+\|z\|_{H^1})\|z\|_{H^1}\|\tilde z\|_{H^1}\big).
\end{aligned}
$$

- To finish with, using integrations by parts and (2.92), we obtain for all $k = 1, \dots, K$,

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^d} \mathrm{Im}(v_k \cdot \nabla\tilde{z}\overline{\tilde{z}})\phi_k \, dx \\
&\quad = -\, 2\, v_k \cdot \mathrm{Im}\int_{\mathbb{R}^d} \tilde{z}_t \nabla\overline{\tilde{z}}\phi_k \, dx - v_k \cdot \mathrm{Im}\int_{\mathbb{R}^d} \tilde{z}_t \overline{\tilde{z}}\nabla\phi_k \, dx + v_k \cdot \mathrm{Im}\int_{\mathbb{R}^d} \nabla\tilde{z}\overline{\tilde{z}}\partial_t\phi_k \, dx \\
&\quad = 2\, v_{k,1}\int_{\mathbb{R}^d} |\nabla\tilde{z}|^2\, \partial_{x_1}\phi_k \, dx - \frac{v_{k,1}}{2}\int_{\mathbb{R}^d} |\tilde{z}|^2\partial_{x_1}^3\phi_k \, dx + v_k \cdot \int_{\mathbb{R}^d} \mathrm{Im}\left(\nabla\tilde{z}\overline{\tilde{z}}\right)\partial_t\phi_k \, dx \qquad (Main_{2,k}) \\
&\qquad + 2\, v_k \cdot \int_{\mathbb{R}^d} \mathrm{Re}\left(\nabla\varphi\overline{\tilde{z}})\big(f(|\tilde{z}+\varphi|^2) - f(|\varphi|^2)\right)\phi_k \, dx \qquad (A_{3,k}) \\
&\qquad + \; v_k \cdot \int_{\mathbb{R}^d} \nabla(f(|\varphi|^2))|\tilde{z}|^2\phi_k \, dx \qquad (B_{2,k}) \\
&\qquad - 2\, v_k \cdot \mathrm{Re}\int_{\mathbb{R}^d} \nabla\left(\varphi\overline{\tilde{z}}\right)\left(f(|\tilde{z}+\varphi|^2) - f(|\varphi|^2)\right)\phi_k \, dx \qquad (B_{3,k}) \\
&\qquad - 2\, v_k \cdot \; \mathrm{Im}\int_{\mathbb{R}^d} \big(ia_k' R_k + b_k' \cdot \nabla R_k\big)\nabla\overline{\tilde{z}}\phi_k \, dx \qquad (C_{4,k}) \\
&\qquad + \mathrm{O}\big((e^{-\gamma t} + \|z\|_{H^1})\|z\|_{H^1}\|\tilde{z}\|_{H^1}\big).
\end{aligned}
$$

We now continue the proof by showing how the corresponding terms put together can yield estimation (2.100).

Step 2: Estimate concerning $\tilde{H}'$

We first deal with the terms A_1, $A_{2,k}$, and $A_{3,k}$ ($k = 1, \dots, K$). We see that

$$
\begin{aligned}
&\left| A_1 - 2\sum_{k=1}^{K} \mathrm{Im}\int_{\mathbb{R}^d} \big(\Delta R_k + f(|R_k|^2)R_k\big)\overline{\tilde{z}}\big(f(|\tilde{z}+\varphi|^2) - f(|\varphi|^2)\big)\, dx \right| \\
&\quad \leq 2\left|\int_{\mathbb{R}^d} \Delta(\varphi - R)\overline{\tilde{z}}\left(f(|\tilde{z}+\varphi|^2) - f(|\varphi|^2)\right)\, dx\right| \\
&\qquad + \int_{\mathbb{R}^d} \big|f(|\varphi|^2) - f(|R|^2)\big|\big|\varphi\overline{\tilde{z}}\big|\big|f(|\tilde{z}+\varphi|^2) - f(|\varphi|^2)\big|\, dx \\
&\qquad + \int_{\mathbb{R}^d} \big|f(|R|^2)(\varphi - R)\overline{\tilde{z}}\big|\big|f(|\tilde{z}+\varphi|^2) - f(|\varphi|^2)\big|\, dx \\
&\qquad + \sum_{k=1}^{K}\int_{\mathbb{R}^d} \left|\left(f(|R|^2) - f(|R_k|^2)\right)R_k\right|\big|\overline{\tilde{z}}\big|\,\big|f(|\tilde{z}+\varphi|^2) - f(|\varphi|^2)\big|\, dx.
\end{aligned}
\tag{2.105}
$$

As for the proof of Lemma 2.18, by the mean value theorem, we observe also that

$$
\begin{cases}
\big|f(|\varphi|^2) - f(|R|^2)\big| \leq C\left(|\varphi - R|^{\frac{4}{d}} + |\varphi - R|\right) \\
\big|\left(f(|R|^2) - f(|R_k|^2)\right)R_k\big| \leq C\displaystyle\sum_{j\neq k} |R_j R_k|.
\end{cases}
\tag{2.106}
$$

Moreover, we deduce from Lemma 2.18 that

$$\begin{aligned}\left|\int_{\mathbb{R}^d} \Delta(\varphi - R)\overline{\tilde{z}}\left(f(|\tilde{z}+\varphi|^2) - f(|\varphi|^2)\right)\, dx\right| &\le Ce^{-\gamma t}\left(\|\tilde{z}\|_{H^1}^{\frac{4}{d}+1} + \|\tilde{z}\|_{H^1}^2\right)\\ &\le Ce^{-\gamma t}\|\tilde{z}\|_{H^1}^2.\end{aligned} \tag{2.107}$$

Let us establish the preceding inequality in each dimension $d = 1, 2, 3$.

- For $d = 1$, we firstly make use of one integration by parts:

$$\begin{aligned}&\int_{\mathbb{R}^d} \Delta(\varphi - R)\overline{\tilde{z}}\left(f(|\tilde{z}+\varphi|^2) - f(|\varphi|^2)\right)\, dx\\ &\quad = -\int_{\mathbb{R}^d} \nabla(\varphi - R)\cdot\nabla\overline{\tilde{z}}\left(f(|\tilde{z}+\varphi|^2) - f(|\varphi|^2)\right)\, dx\\ &\quad\quad - \int_{\mathbb{R}^d} \nabla(\varphi - R)\cdot\nabla\left(f(|\tilde{z}+\varphi|^2) - f(|\varphi|^2)\right)\overline{\tilde{z}}\, dx.\end{aligned}$$

We thus obtain by Lemma 2.18:

$$\begin{aligned}&\left|\int_{\mathbb{R}^d} \Delta(\varphi - R)\overline{\tilde{z}}\left(f(|\tilde{z}+\varphi|^2) - f(|\varphi|^2)\right)\, dx\right|\\ &\quad \le C\int_{\mathbb{R}^d} |\nabla(\varphi - R)||\nabla\tilde{z}|(|z| + |z|^{\frac{4}{d}}) + C\int_{\mathbb{R}^d} |\nabla(\varphi - R)||\nabla\varphi|(|z| + |z|^{\frac{4}{d}})\, dx.\end{aligned} \tag{2.108}$$

Then we note that for all $\psi \in H^1(\mathbb{R}^d)$, (by the embedding $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$)

$$\begin{aligned}\int_{\mathbb{R}^d} |\nabla\psi||\nabla\tilde{z}||\tilde{z}|\, dx &\le \|\nabla\tilde{z}\|_{L^2}\|\nabla\psi\|_{L^2}\|\tilde{z}\|_{L^\infty} \le C\|\tilde{z}\|_{H^1}^2\|\psi\|_{H^1},\\ \int_{\mathbb{R}^d} |\nabla\tilde{z}||\nabla\varphi||\tilde{z}|^{\frac{4}{d}}\, dx &\le \|\tilde{z}\|_{H^1}^{\frac{4}{d}}\|\nabla\tilde{z}\|_{L^2}\|\nabla\varphi\|_{L^2} \le C\|\tilde{z}\|_{H^1}^{\frac{4}{d}+1}\|\varphi\|_{H^1},\\ \int_{\mathbb{R}^d} |\nabla\psi||\nabla\varphi||\tilde{z}|^{\frac{4}{d}}\, dx &\le \|\tilde{z}\|_{H^1}^{\frac{4}{d}}\|\nabla\psi\|_{L^2}\|\nabla\varphi\|_{L^2} \le C\|\tilde{z}\|_{H^1}^{\frac{4}{d}}\|\psi\|_{H^1}\|\varphi\|_{H^1}.\end{aligned}$$

- For $d = 2$: for all $\psi \in H^2(\mathbb{R}^d)$, (by the embeddings $H^1(\mathbb{R}^2) \hookrightarrow L^q(\mathbb{R}^2)$ for each $q \in [2, +\infty)$)

$$\begin{aligned}\int_{\mathbb{R}^d} |\nabla\psi||\nabla\tilde{z}||\tilde{z}|\, dx &\le \|\nabla\tilde{z}\|_{L^2}\|\nabla\psi\|_{L^4}\|\tilde{z}\|_{L^4} \le C\|\tilde{z}\|_{H^1}^2\|\psi\|_{H^2},\\ \int_{\mathbb{R}^d} |\nabla\psi||\nabla\tilde{z}||\tilde{z}|^{\frac{4}{d}}\, dx &\le \|\nabla\tilde{z}\|_{L^2}\|\tilde{z}\|_{L^{\frac{16}{d}}}^{\frac{4}{d}}\|\nabla\psi\|_{L^4} \le C\|\tilde{z}\|_{H^1}^{\frac{4}{d}+1}\|\psi\|_{H^2}\\ \int_{\mathbb{R}^d} |\nabla\psi||\nabla\varphi||\tilde{z}|^{\frac{4}{d}}\, dx &\le \|\tilde{z}\|_{L^{\frac{8}{d}}}^{\frac{4}{d}}\|\nabla\psi\|_{L^4}\|\nabla\varphi\|_{L^4} \le C\|\tilde{z}\|_{H^1}^{\frac{4}{d}}\|\psi\|_{H^2}\|\varphi\|_{H^2}.\end{aligned}$$

- For $d = 3$, $|z| + |z|^{\frac{4}{d}} \le |z| + |z|^{\frac{4}{3}} \le 2(|z| + |z|^2)$. We have for all $\psi \in H^2(\mathbb{R}^d)$, (by the embedding $H^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$)

$$\int_{\mathbb{R}^d} |\nabla\psi||\nabla\tilde{z}||\tilde{z}|\, dx \le \|\nabla\tilde{z}\|_{L^2}\|\nabla\psi\|_{L^4}\|\tilde{z}\|_{L^4} \le C\|\tilde{z}\|_{H^1}^2\|\psi\|_{H^2},$$

$$\int_{\mathbb{R}^d} |\nabla\psi||\nabla\tilde{z}||\tilde{z}|^2 \, dx \leq \|\nabla\tilde{z}\|_{L^2}\|\tilde{z}\|_{L^6}^2\|\nabla\psi\|_{L^6} \leq C\|\tilde{z}\|_{H^1}^3\|\psi\|_{H^2}$$
$$\int_{\mathbb{R}^d} |\nabla\psi||\nabla\varphi||\tilde{z}|^2 \, dx \leq \|\nabla\psi\|_{L^2}\|\tilde{z}\|_{L^6}^3\|\nabla\varphi\|_{L^6} \leq C\|\tilde{z}\|_{H^1}^2\|\psi\|_{H^2}\|\varphi\|_{H^2}.$$

Hence, gathering (2.105), (2.106), and(2.107), using (2.74) and the fact that $\frac{4}{d} + 1 \geq 2$, it results

$$\left| A_1 - 2\sum_{k=1}^{K} \mathrm{Im} \int_{\mathbb{R}^d} \left(\Delta R_k + f(|R_k|^2)R_k\right)\overline{\tilde{z}}\left(f(|\tilde{z}+\varphi|^2) - f(|\varphi|^2)\right) dx \right| \leq Ce^{-\gamma t}\left(\|\tilde{z}\|_{H^1}^2\right). \tag{2.109}$$

In a similar way, for all $k = 1, \dots, K$,

$$\begin{aligned}
&\left| A_{2,k} + 2\left(\omega_k + \frac{|v_k|^2}{4}\right)\int_{\mathbb{R}^d} \mathrm{Im}(R_k\overline{\tilde{z}})\left(f(|\tilde{z}+\varphi|^2) - f(|\varphi|^2)\right)\phi_k \, dx \right| \\
&\qquad \leq 2\left(\omega_k + \frac{|v_k|^2}{4}\right)\int_{\mathbb{R}^d} \left|(\varphi - R)\overline{\tilde{z}}\left(f(|\tilde{z}+\varphi|^2) - f(|\varphi|^2)\right)\phi_k\right| dx \\
&\qquad\quad + 2\left(\omega_k + \frac{|v_k|^2}{4}\right)\sum_{j\neq k}\int_{\mathbb{R}^d} \left|R_j\overline{\tilde{z}}\left(f(|\tilde{z}+\varphi|^2) - f(|\varphi|^2)\right)\phi_k\right| dx \\
&\qquad \leq C\int_{\mathbb{R}^d} \left|\varphi - R\right|\left(|\tilde{z}|^{\frac{4}{d}+1} + |\tilde{z}|^2\right) dx + C\sum_{j\neq k}\int_{\mathbb{R}^d} \left|R_j\phi_k\right|\left(|\tilde{z}|^{\frac{4}{d}+1} + |\tilde{z}|^2\right) dx \\
&\qquad \leq Ce^{-\gamma t}\|\tilde{z}\|_{H^1}^2.
\end{aligned} \tag{2.110}$$

We have then for all $k = 1, \dots, K$,

$$\begin{aligned}
&\left| A_{3,k} - 2v_k \cdot \int_{\mathbb{R}^d} \mathrm{Re}(\nabla R_k\overline{\tilde{z}})\left(f(|\tilde{z}+\varphi|^2) - f(|\varphi|^2)\right)\phi_k \, dx \right| \\
&\qquad \leq C\int_{\mathbb{R}^d} |\nabla(\varphi - R)|\left(|\tilde{z}|^p + |\tilde{z}|^2\right)\phi_k \, dx + C\sum_{j\neq k}\int_{\mathbb{R}^d} \left|\nabla R_j\phi_k\right|\left(|\tilde{z}|^{\frac{4}{d}+1} + |\tilde{z}|^2\right) dx \\
&\qquad \leq Ce^{-\gamma t}\|\tilde{z}\|_{H^1}^2.
\end{aligned} \tag{2.111}$$

Let us gather (2.109), (2.110), and (2.111). Observing that

$$\begin{cases} \partial_t R_k = -v_k \cdot \nabla R_k + i\left(\omega_k + \frac{|v_k|^2}{4}\right)R_k \\ \partial_t R_k = i\left(\Delta R_k + f(|R_k|^2)R_k\right), \end{cases} \tag{2.112}$$

we notice that

$$\mathrm{Im}\left((\Delta R_k + f(|R_k|^2)R_k)\overline{\tilde{z}}\right) - \left(\omega_k + \frac{|v_k|^2}{4}\right)\mathrm{Im}\left(R_k\phi_k\overline{\tilde{z}}\right) - v_k\cdot\mathrm{Re}\left(\nabla R_k\phi_k\overline{\tilde{z}}\right) = -\mathrm{Re}\left(\partial_t R_k(1-\phi_k)\overline{\tilde{z}}\right).$$

As a consequence of (2.90), we obtain a control of

$$\int_{\mathbb{R}^d} \left\{ 2\mathrm{Im}\left((\Delta R_k + f(|R_k|^2)R_k)\overline{\tilde{z}}\right) - 2\left(\omega_k + \frac{|v_k|^2}{4}\right)\mathrm{Im}\left(R_k\phi_k\overline{\tilde{z}}\right) \right.$$

$$-2v_k\cdot\mathrm{Re}\left(\nabla R_k\phi_k\overline{\tilde z}\right)\left(f(|\tilde z+\varphi|^2)-f(|\varphi|^2)\right)\Big\}\,dx$$

by $Ce^{-\gamma t}\|\tilde z\|^2_{H^1}$.
Finally,

$$\left|A_1+\sum_{k=1}^{K}\left\{A_{2,k}-A_{3,k}\right\}\right|\le Ce^{-\gamma t}\|\tilde z\|^2_{H^1}. \tag{2.113}$$

Let us focus now on the terms identified by the letter B. We observe that

$$B_{2,k}=-v_k\cdot\int_{\mathbb{R}^d}f(|\tilde z+\varphi|^2)\nabla(|\tilde z|^2)\phi_k\,dx+\mathrm{O}\left((e^{-\gamma t}+\|z\|_{H^1})\|z\|_{H^1}\|\tilde z\|_{H^1}\right). \tag{2.114}$$

Then, we obtain

$$\begin{aligned}B_{2,k}+B_{3,k}=&-v_k\cdot\mathrm{Re}\int_{\mathbb{R}^d}\nabla\left(|\tilde z+\overline{\varphi}|^2\right)f(|\tilde z+\overline{\varphi}|^2)\phi_k\,dx+v_k\cdot\mathrm{Re}\int_{\mathbb{R}^d}\nabla\left(|\varphi|^2\right)f(|\tilde z+\overline{\varphi}|^2)\phi_k\,dx\\&+2v_k\cdot\mathrm{Re}\int_{\mathbb{R}^d}\nabla(\varphi\overline{\tilde z})f(|\varphi|^2)\phi_k\,dx+\mathrm{O}\left((e^{-\gamma t}+\|z\|_{H^1})\|z\|_{H^1}\|\tilde z\|_{H^1}\right).\end{aligned} \tag{2.115}$$

Notice next that

$$v_k\cdot\mathrm{Re}\left(\nabla R_k\overline{R_k}\right)=\mathrm{Im}\left(\Delta R_k\overline{R_k}\right),$$

which allows us to rewrite

$$B_1-2\sum_{k=1}^{K}v_k\cdot\mathrm{Re}\int_{\mathbb{R}^d}\nabla R_k\overline{R_k}\left(f(|\tilde z+\varphi|^2)-f(|\varphi|^2)-2\mathrm{Re}(\tilde z\overline{\varphi})f'(|\varphi|^2)\right)dx$$

as a sum of quantities in which the differences between φ and R or the products $\nabla R_k\overline{R_j}$ for $j\neq k$ appear. Use moreover

1. on the one hand, the second estimate proven in Lemma 2.18
2. on the other, the following inequalities:

   - for $d=1$: for all $\psi\in H^1(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d}|\nabla\psi||\nabla\tilde z||\tilde z|\,dx\le\|\nabla\tilde z\|_{L^2}\|\nabla\psi\|_{L^2}\|\tilde z\|_{L^\infty}\le C\|\tilde z\|^2_{H^1}\|\psi\|_{H^1},$$

$$\int_{\mathbb{R}^d}|\nabla\psi|^2|\tilde z|^2\,dx\le\|\tilde z\|^2_{H^1}\|\nabla\psi\|^2_{L^2}\le C\|\tilde z\|^2_{H^1}\|\psi\|^2_{H^1},$$

$$\int_{\mathbb{R}^d}|\nabla\psi||\nabla\varphi||\tilde z|^{\frac{4}{d}}\,dx\le\|\tilde z\|^{\frac{4}{d}}_{H^1}\|\nabla\psi\|_{L^2}\|\nabla\varphi\|_{L^2}\le C\|\tilde z\|^{\frac{4}{d}}_{H^1}\|\psi\|_{H^1}\|\varphi\|_{H^1},$$

   - for $d\in\{2,3\}$: for all $\psi\in H^2(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d}|\nabla\psi||\nabla\tilde z||\tilde z|\,dx\le\|\nabla\tilde z\|_{L^2}\|\nabla\psi\|_{L^4}\|\tilde z\|_{L^4}\le C\|\tilde z\|^2_{H^1}\|\psi\|_{H^2},$$

$$\int_{\mathbb{R}^d} |\nabla\psi|^2|\tilde{z}|^2 \, dx \leq \|\tilde{z}\|_{L^4}^2\|\nabla\psi\|_{L^4}^2 \leq C\|\tilde{z}\|_{H^1}^2\|\psi\|_{H^2}^2,$$
$$\int_{\mathbb{R}^d} |\nabla\psi||\nabla\varphi||\tilde{z}|^4 \, dx \leq \|\tilde{z}\|_{L^6}^4\|\nabla\psi\|_{L^6}\|\nabla\varphi\|_{L^6} \leq C\|\tilde{z}\|_{H^1}^4\|\psi\|_{H^2}\|\varphi\|_{H^2}.$$

Remark 2.8. Regarding the higher dimensions in order to prove Proposition 2.6, one would make use of the following inequality, valid for $d \geq 3$: for all $\psi \in H^{\lfloor \frac{d}{2} \rfloor+1}(\mathbb{R}^d)$, for all $\tilde{\psi} \in H^1(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} |\nabla\psi||\nabla\tilde{\psi}||\tilde{z}| \, dx \leq \|\tilde{z}\|_{L^{\frac{2d}{d-2}}}\|\nabla\tilde{\psi}\|_{L^2}\|\nabla\psi\|_{L^d} \leq C\|\tilde{z}\|_{H^1}\|\tilde{\psi}\|_{H^2}\|\psi\|_{H^{\lfloor \frac{d}{2} \rfloor+1}}.$$

Then we conclude that

$$\begin{aligned}&\left|B_1 - 2\sum_{k=1}^K v_k \cdot \mathrm{Re}\int_{\mathbb{R}^d} \nabla R_k\overline{R_k}\big(f(|\tilde{z}+\varphi|^2) - f(|\varphi|^2) - 2\mathrm{Re}(\tilde{z}\overline{\varphi})f'(|\varphi|^2)\big)\,dx\right| \\ &\quad\leq Ce^{-\gamma t}\big(\|\tilde{z}\|_{H^1}^{\frac{4}{d}+1} + \|\tilde{z}\|_{H^1}^2\big) \leq Ce^{-\gamma t}\|\tilde{z}\|_{H^1}^2.\end{aligned} \tag{2.116}$$

Thus, from (2.115) and (2.116), we deduce that:

$$\left|B_1 - \sum_{k=1}^K \big(B_{2,k} + B_{3,k}\big)\right| \tag{2.117}$$
$$\begin{aligned}&\leq Ce^{-\gamma t}\|\tilde{z}\|_{H^1}^2 + C\left((e^{-\gamma t} + \|z\|_{H^1})\|z\|_{H^1}\|\tilde{z}\|_{H^1}\right) \\ &\quad+ \sum_{k=1}^K \left|v_k \cdot \mathrm{Re}\int_{\mathbb{R}^d} \nabla\left(|R_k|^2 - |\varphi|^2\right) f(|\tilde{z}+\varphi|^2)\phi_k \, dx\right| \\ &\quad+ \sum_{k=1}^K \left|-v_k \cdot \int_{\mathbb{R}^d} \nabla\left(|\varphi|^2\right) f(|\varphi|^2) \, dx + v_k \cdot \mathrm{Re}\int_{\mathbb{R}^d} \nabla(|\tilde{z}+\varphi|^2)f(|\tilde{z}+\varphi|^2) \, dx\right| \\ &\leq C\left(e^{-\gamma t} + \|z\|_{H^1}\right)\|z\|_{H^1}\|\tilde{z}\|_{H^1}.\end{aligned} \tag{2.118}$$

Notice that we have used (2.87),

$$\int_{\mathbb{R}^d} \nabla\left(|\varphi|^2\right) f(|\varphi|^2) \, dx = 0, \quad \text{and} \quad \int_{\mathbb{R}^d} \nabla\left(|\tilde{z}+\varphi|^2\right) f(|\tilde{z}+\varphi|^2) \, dx = 0.$$

To finish with, we have to obtain estimates for the terms with the letter C involving a_k' and b_k'. Due to (2.81), (2.90), and (2.112), we compute

$$\begin{aligned}&-2\sum_{k=1}^K a_k' \int_{\mathbb{R}^d} \left\{\mathrm{Re}\left(iR_k\Delta\overline{\tilde{z}}\right) + \mathrm{Re}\left(iR_k\overline{\tilde{z}}\right) - \left(\omega_k + \frac{|v_k|^2}{4}\right)\mathrm{Re}\left(iR_k\overline{\tilde{z}}\phi_k\right) - v_k \cdot \mathrm{Im}\left(iR_k\nabla\overline{\tilde{z}}\phi_k\right)\right\} dx \\ &\quad= 2\sum_{k=1}^K a_k' \,\mathrm{Im}\int_{\mathbb{R}^d} \left\{\overline{\tilde{z}}\left(\Delta R_k + R_k f\left(|R_k|^2\right) - \left(\omega_k + \frac{|v_k|^2}{4}\right)R_k\phi_k - iv_k \cdot \nabla R_k\phi_k\right)\right\} dx \\ &\quad\quad+ \mathrm{O}\big(e^{-\gamma t}\|\tilde{z}\|_{H^1}\|z\|_{H^1}\big) \\ &\quad= \mathrm{O}\left(e^{-\gamma t}\|\tilde{z}\|_{H^1}\|z\|_{H^1}\right).\end{aligned} \tag{2.119}$$

On the other hand,

$$\begin{aligned}
&-2\sum_{k=1}^{K} b_k' \cdot \int_{\mathbb{R}^d} \Big\{ \mathrm{Re}(\nabla R_k \Delta\bar{\tilde{z}}) + \mathrm{Re}(\nabla R_k \bar{\tilde{z}} f(|\tilde{z}+\varphi|^2)) + \mathrm{Re}\big(\nabla R_k \overline{R_k}\big(f(|\tilde{z}+\varphi|^2) - f(|\varphi|^2)\big)\big) \\
&\qquad - \left(\omega_k + \frac{|v_k|^2}{4}\right) \mathrm{Re}\left(\nabla R_k \bar{\tilde{z}} \phi_k\right) - \mathrm{Im}\left(v_k \cdot R_k \nabla \bar{\tilde{z}} \phi_k\right) \Big\}\, dx \\
&= 2\sum_{k=1}^{K} b_k' \cdot \mathrm{Re} \int_{\mathbb{R}^d} \left\{ \nabla \bar{\tilde{z}} \left(\Delta R_k + R_k f\left(|R_k|^2\right) - \left(\omega_k + \frac{|v_k|^2}{4}\right) R_k \phi_k - i v_k \cdot \nabla R_k \phi_k \right) \right\} dx \\
&\quad + 2\sum_{k=1}^{K} b_k' \cdot \int_{\mathbb{R}^d} \left\{ \mathrm{Re}\left(R_k \bar{\tilde{z}} \nabla \left(f\left(|R_k|^2\right)\right)\right) - \mathrm{Re}\left(\nabla R_k \overline{R_k}\right) \left(f\left(|\tilde{z}+\varphi|^2\right) - f\left(|\varphi|^2\right)\right) \right\} dx \\
&\quad + \mathrm{O}\left((e^{-\gamma t} + \|\tilde{z}\|_{H^1})\|\tilde{z}\|_{H^1}\|z\|_{H^1}\right) \\
&= \mathrm{O}\left((e^{-\gamma t} + \|\tilde{z}\|_{H^1})\|\tilde{z}\|_{H^1}\|z\|_{H^1}\right),
\end{aligned} \tag{2.120}$$

again due to (2.81), (2.90), and (2.112). Consequently, (2.119) and (2.120) lead to

$$C_1 + C_2 + \sum_{k=1}^{K} \{C_{3,k} - C_{4,k}\} = \mathrm{O}\big(\left(e^{-\gamma t} + \|\tilde{z}\|_{H^1}\right) \|\tilde{z}\|_{H^1} \|z\|_{H^1} \big). \tag{2.121}$$

Proposition 2.17 follows from Step 1, from estimates (2.113), (2.117), and (2.121), and from the observation that

$$Main = \sum_{k=1}^{K} \left(Main_{1,k} - Main_{2,k}\right) = \mathrm{O}\left(\frac{1}{t}\|\tilde{z}(t)\|_{H^1}^2\right)$$

by (2.91). □

Control of the R_k directions

We have the following estimate which expresses that the variation in time of the real scalar products $\mathrm{Re}\int_{\mathbb{R}^d} \tilde{z}(t)\overline{R_k}(t)\, dx$ (which appear in (2.94)) is essentially of order two in $z(t)$.

Lemma 2.20. *For all $t \geq T_1$,*

$$\left| \frac{d}{dt} \mathrm{Re} \int_{\mathbb{R}^d} \tilde{z}(t) \overline{R_k}(t)\, dx \right| \leq C \left(e^{-\gamma t} \|z(t)\|_{H^1} + \|z(t)\|_{H^1}^2 \right). \tag{2.122}$$

Proof. We notice that

$$\begin{aligned}
\mathrm{Re} \int_{\mathbb{R}^d} \tilde{z}\overline{R_k}\, dx &= \mathrm{Re} \int_{\mathbb{R}^d} z\overline{R_k}\, dx + \sum_{j=1}^{K} \mathrm{Re} \int_{\mathbb{R}^d} i a_j R_j \overline{R_k}\, dx + \sum_{j=1}^{K} \mathrm{Re} \int_{\mathbb{R}^d} b_j \cdot \nabla R_j \overline{R_k}\, dx \\
&= \mathrm{Re} \int_{\mathbb{R}^d} z\overline{R_k}\, dx + a_k \mathrm{Re} \int_{\mathbb{R}^d} i|R_k|^2\, dx + b_k \cdot \mathrm{Re} \int_{\mathbb{R}^d} \nabla R_k \overline{R_k}\, dx + \eta(t) \\
&= \mathrm{Re} \int_{\mathbb{R}^d} z\overline{R_k}\, dx + \eta(t),
\end{aligned} \tag{2.123}$$

where η is a complex-valued function defined on a neighborhood of $+\infty$, differentiable in the sense of distributions, and such that $\eta'(t) = \mathrm{O}\,(e^{-\gamma t}\|z(t)\|_{H^1})$. Moreover,

$$\begin{aligned}\frac{d}{dt}\mathrm{Re}\int_{\mathbb{R}^d} z\overline{R_k}\,dx &= \mathrm{Re}\int_{\mathbb{R}^d} i\left(\Delta z + f\left(|z+\varphi|^2\right)z + \left(f\left(|z+\varphi|^2\right) - f\left(|\varphi|^2\right)\right)\varphi\right)\overline{R_k}\,dx \\ &\quad + \mathrm{Re}\int_{\mathbb{R}^d} -iz\left(\Delta\overline{R_k} + f\left(|R_k|^2\right)\overline{R_k}\right)\,dx \\ &= \mathrm{Re}\int_{\mathbb{R}^d} i\left(f\left(|\varphi|^2\right) - f\left(|R_k|^2\right)\right)z\overline{R_k}\,dx \\ &\quad + \mathrm{Re}\int_{\mathbb{R}^d} i\left(f\left(|z+\varphi|^2\right) - f\left(|\varphi|^2\right)\right)\varphi\overline{R_k}\,dx + \mathrm{O}\left(\|z(t)\|_{H^1}^2\right),\end{aligned} \tag{2.124}$$

where we have used $p \geq 2$ and

$$\int_{\mathbb{R}^d} \Delta z\overline{R_k}\,dx = \int_{\mathbb{R}^d} z\Delta\overline{R_k}\,dx \qquad \text{and} \qquad \left|f(|z+\varphi|^2) - f(|\varphi|^2)\right| \leq C(|z| + |z|^{\frac{4}{d}}).$$

By means of

$$\left|f(|\varphi|^2) - f(|R_k|^2)\right| \leq C\big(|\varphi - R|^2 + |\varphi - R| + \sum_{j\neq k}|R_jR_k|\big)$$

(which is consequence of the application of the mean value theorem, as for the proof of Lemma 2.18), and by means of (2.24) and (2.74), we see that

$$\left|\mathrm{Re}\int_{\mathbb{R}^d} i\left(f(|\varphi|^2) - f(|R_k|^2)\right)z\overline{R_k}\,dx\right| \leq Ce^{-\gamma t}\|z(t)\|_{L^2}. \tag{2.125}$$

Similarly by Lemma 2.18, and noting in addition that

$$\mathrm{Re}\left(i\varphi\overline{R_k}\right) = \mathrm{Re}\left(i((\varphi - R) + (R - R_k))\overline{R_k}\right),$$

we have also

$$\left|\mathrm{Re}\int_{\mathbb{R}^d} i\left(f(|z+\varphi|^2) - f(|\varphi|^2)\right)\varphi\overline{R_k}\,dx\right| \leq Ce^{-\gamma t}\|z(t)\|_{H^1}. \tag{2.126}$$

To put it in a nutshell, Lemma 2.20 is now a direct consequence of (2.123), (2.124), (2.125), and (2.126). □

As a consequence of the preceding lemma, we state:

Corollary 2.21. *For all $t \geq T_1$,*

$$\left|\frac{d}{dt}\left(\mathrm{Re}\int_{\mathbb{R}^d}\tilde{z}(t)\overline{R_k}(t)\,dx\right)^2\right| \leq C\left(e^{-\gamma t}\|z(t)\|_{H^1} + \|z(t)\|_{H^1}^2\right)\|\tilde{z}(t)\|_{H^1}. \tag{2.127}$$

Control of the modulation parameters

At this point, recalling inequality (2.86), it remains us to obtain estimates for $|a_k(t)|$ and $|b_k(t)|$. This is the object of the following result.

Lemma 2.22. *For all $t \geq T_1$,*

$$|a_k'(t)| + |b_k'(t)| \leq C\big(e^{-\gamma t}\|z(t)\|_{H^1} + \|z(t)\|_{H^1}^2 + \|z\|_{H^1}^{\frac{4}{d}} + \|\tilde{z}(t)\|_{L^2}\big). \tag{2.128}$$

Proof. Due to

$$\begin{cases} \Delta\tilde{z} = \Delta z + \sum_{j=1}^{K}\big(ia_j\Delta R_j + b_j\cdot\nabla(\Delta R_j)\big) \\ \partial_t\tilde{z} = \partial_t z + \sum_{k=1}^{K}\big(ia_k' R_k + b_k'\cdot\nabla R_k\big) + \sum_{k=1}^{K}\big(ia_k\partial_t R_k + b_k\cdot\nabla\partial_t R_k\big) \\ \partial_t z = i\big(\Delta z + f(|z+\varphi|^2)z + \big(f(|z+\varphi|^2) - f(|\varphi|^2)\big)\varphi\big), \end{cases}$$

differentiation with respect to t of equality $\operatorname{Im}\int_{\mathbb{R}^d}\tilde{z}\overline{R_k}\,dx = 0$ (2.79) implies:

$$\begin{aligned} 0 &= \operatorname{Im}\int_{\mathbb{R}^d}\partial_t\tilde{z}\overline{R_k}\,dx + \operatorname{Im}\int_{\mathbb{R}^d}\tilde{z}\partial_t\overline{R_k}\,dx \\ &= \operatorname{Re}\int_{\mathbb{R}^d}\big(\Delta z + f(|z+\varphi|^2)z + \big(f(|z+\varphi|^2) - f(|\varphi|^2)\big)\varphi\big)\overline{R_k}\,dx \\ &\quad+ \sum_{j=1}^{K}\operatorname{Im}\left(ia_j'\int_{\mathbb{R}^d}R_j\overline{R_k}\,dx + b_j'\cdot\int_{\mathbb{R}^d}\nabla R_j\overline{R_k}\,dx + ia_j\int_{\mathbb{R}^d}\partial_t R_j\overline{R_k}\,dx + b_j\cdot\int_{\mathbb{R}^d}\nabla\partial_t R_j\overline{R_k}\,dx\right) \\ &\quad- \operatorname{Re}\int_{\mathbb{R}^d}\left(z + \sum_{j=1}^{K}\big(ia_jR_j + b_j\cdot\nabla R_j\big)\right)\big(\Delta\overline{R_k} + f(|R_k|^2)\overline{R_k}\big)\,dx, \end{aligned}$$

or equivalently

$$\begin{aligned} 0 &= \mathrm{O}\left(\|z(t)\|_{H^1}^2\right) + \operatorname{Re}\int_{\mathbb{R}^d}\big(f(|\varphi|^2) - f(|R_k|^2)\big)z\overline{R_k}\,dx + \operatorname{Re}\int_{\mathbb{R}^d}\big(f(|z+\varphi|^2) - f(|\varphi|^2)\big)\varphi\overline{R_k}\,dx \\ &\quad+ \sum_{j=1}^{K}\left(a_j'\operatorname{Re}\int_{\mathbb{R}^d}R_j\overline{R_k}\,dx + b_j'\cdot\operatorname{Im}\int_{\mathbb{R}^d}\nabla R_j\overline{R_k}\,dx\right) \\ &\quad+ \sum_{j=1}^{K}\left(-a_j\operatorname{Im}\int_{\mathbb{R}^d}\big(\Delta R_j + f(|R_j|^2)R_j\big)\overline{R_k}\,dx + b_j\cdot\operatorname{Re}\int_{\mathbb{R}^d}\big(\Delta R_j + f(|R_j|^2)R_j\big)\nabla\overline{R_k}\,dx\right) \\ &\quad- \sum_{j=1}^{K}\left(-a_j\operatorname{Im}\int_{\mathbb{R}^d}R_j\left(\Delta\overline{R_k} + f(|R_k|^2)\overline{R_k}\right)\,dx + b_j\cdot\operatorname{Re}\int_{\mathbb{R}^d}\nabla R_j\left(\Delta\overline{R_k} + f(|R_k|^2)\overline{R_k}\right)\,dx\right) \\ &= \int_{\mathbb{R}^d}\big(f(|z+\varphi|^2) - f(|\varphi|^2)\big)|R_k|^2\,dx + a_k'\int_{\mathbb{R}^d}Q_{\omega_k}^2\,dx + \frac{1}{2}b_k'\cdot v_k\int_{\mathbb{R}^d}Q_{\omega_k}^2\,dx \\ &\quad+ \mathrm{O}\left(\|z(t)\|_{H^1}^2 + e^{-\gamma t}\|z(t)\|_{H^1}\right). \end{aligned} \tag{2.129}$$

Similarly, exploiting the d-dimensional equality $\operatorname{Re}\int_{\mathbb{R}^d}\tilde{z}\nabla\overline{R_k}\,dx = 0$ (2.79), we see that

$$0 = \operatorname{Im}\int_{\mathbb{R}^d}\left(\Delta\tilde{z} - \sum_{j=1}^{K}ia_j\Delta R_j - \sum_{j=1}^{K}b_j\cdot\nabla\Delta R_j\right)\nabla\overline{R_k}\,dx + \mathrm{O}\left(\|z(t)\|_{L^2}^2\right)$$

$$
\begin{aligned}
&-\operatorname{Im}\int_{\mathbb{R}^d} f(|\varphi|^2)\left(\tilde{z}-\sum_{j=1}^K ia_jR_j-\sum_{j=1}^K b_j\cdot\nabla R_j\right)\nabla\overline{R_k}\,dx\\
&-\operatorname{Im}\int_{\mathbb{R}^d}\big(f(|z+\varphi|^2)-f(|\varphi|^2)\big)\varphi\nabla\overline{R_k}\,dx\\
&+\sum_{j=1}^K\left[a_j'\operatorname{Im}\int_{\mathbb{R}^d}\nabla R_k\overline{R_j}\,dx+b_j'\cdot\operatorname{Re}\int_{\mathbb{R}^d}\nabla R_j\nabla\overline{R_k}\right]\\
&-\sum_{j=1}^K a_j\operatorname{Re}\int_{\mathbb{R}^d}\Delta R_j\nabla\overline{R_k}\,dx-\sum_{j=1}^K a_j\operatorname{Re}\int_{\mathbb{R}^d} f(|R_j|^2)R_j\nabla\overline{R_k}\,dx\\
&-\sum_{j=1}^K b_j\cdot\operatorname{Im}\int_{\mathbb{R}^d}\nabla(\Delta R_j)\nabla\overline{R_k}\,dx-\sum_{j=1}^K b_j\cdot\operatorname{Im}\int_{\mathbb{R}^d}\nabla\big(f(|R_j|^2)R_j\big)\nabla\overline{R_k}\,dx\\
&+\operatorname{Im}\int_{\mathbb{R}^d}\tilde{z}\big(\nabla(\Delta\overline{R_k})+\nabla\big(f(|R_k|^2)\overline{R_k}\big)\big)\,dx,
\end{aligned}
$$

or equivalently, using

$$
\begin{aligned}
\int_{\mathbb{R}^d}\nabla\big(f(Q_{\omega_k}^2)\big)Q_{\omega_k}^2\,dx&=-\sum_{i=1}^d\int_{\mathbb{R}^d} f(Q_{\omega_k}^2)\partial_{x_i}(Q_{\omega_k}^2)\,dx\\
&=0,
\end{aligned}
$$

$$
\begin{aligned}
0&=\mathrm{O}\left(\|z(t)\|_{H^1}^2+e^{-\gamma t}\|z(t)\|_{H^1}\right)+\int_{\mathbb{R}^d}\big(f(|z+\varphi|^2)-f(|\varphi|^2)\big)\operatorname{Im}\big(\nabla R_k\overline{R_k}\big)\,dx\\
&\quad+\operatorname{Im}\int_{\mathbb{R}^d}\tilde{z}\nabla(f(|R_k|^2))\overline{R_k}\,dx+\frac{a_k'}{2}v_k\int_{\mathbb{R}^d}Q_{\omega_k}^2\,dx+\operatorname{Re}\int_{\mathbb{R}^d}\left[\nabla\overline{R_k}^t\nabla R_k\right]\times b_k'\,dx\\
&=\mathrm{O}\left(\|z(t)\|_{H^1}^2+e^{-\gamma t}\|z(t)\|_{H^1}\right)+\int_{\mathbb{R}^d}\big(f(|z+\varphi|^2)-f(|\varphi|^2)\big)\operatorname{Im}\big(\nabla R_k\overline{R_k}\big)\,dx \qquad (2.130)\\
&\quad+\operatorname{Im}\int_{\mathbb{R}^d}\tilde{z}\nabla\left(f(|R_k|^2)\right)\overline{R_k}\,dx+\frac{a_k'}{2}v_k\int_{\mathbb{R}^d}Q_{\omega_k}^2\,dx\\
&\quad+\int_{\mathbb{R}^d}\left[\nabla Q_{\omega_k}{}^t\nabla Q_{\omega_k}+\frac14 v_k{}^t v_kQ_{\omega_k}^2\right]\times b_k'\,dx,
\end{aligned}
$$

Using Lemma 2.18 and (2.24), we obtain

$$
\begin{aligned}
&\int_{\mathbb{R}^d}\big(f(|z+\varphi|)^2-f(|\varphi|^2)\big)\operatorname{Im}\big(\nabla R_k\overline{R_k}\big)\,dx\\
&\quad=2\int_{\mathbb{R}^d}\operatorname{Re}(z\overline{\varphi})f'(|\varphi|^2)\operatorname{Im}\big(\nabla R_k\overline{R_k}\big)\,dx+\mathrm{O}\left(\|z\|_{H^1}^2+\|z\|_{H^1}^{\frac4d}\right)\\
&\quad=2\int_{\mathbb{R}^d}\operatorname{Re}(z\overline{R_k})f'(|R_k|^2)\operatorname{Im}\big(\nabla R_k\overline{R_k}\big)\,dx+\mathrm{O}\left(e^{-\gamma t}\|z\|_{H^1}+\|z\|_{H^1}^2+\|z\|_{H^1}^{\frac4d}\right).
\end{aligned}
$$

Recalling the definition of $\tilde{z}$ (2.78), this reads also as follows:

$$
\int_{\mathbb{R}^d}\big(f(|z+\varphi|)^2-f(|\varphi|^2)\big)\operatorname{Im}\left(\nabla R_k\overline{R_k}\right)\,dx
$$

$$= 2\int_{\mathbb{R}^d} \mathrm{Re}\left(\tilde{z}\overline{R_k}\right) f'(|R_k|^2)\mathrm{Im}\left(\nabla R_k \overline{R_k}\right)\, dx + \mathrm{O}\left(e^{-\gamma t}\|z\|_{H^1} + \|z\|_{H^1}^2\right)$$
$$= \mathrm{O}\left(\|\tilde{z}\|_{L^2} + e^{-\gamma t}\|z\|_{H^1} + \|z\|_{H^1}^2 + \|z\|_{H^1}^{\frac{4}{d}}\right).$$

Considering that $\mathrm{Im}\left(\nabla R_k \overline{R_k}\right) = v_k |R_k|^2$, we then deduce from (3.99) and (3.100) that

$$\int_{\mathbb{R}^d} \left[\nabla Q_{\omega_k} {}^t\nabla Q_{\omega_k}\right]\, dx \times b_k' = \mathrm{O}\left(\|\tilde{z}\|_{L^2} + e^{-\gamma t}\|z\|_{H^1} + \|z\|_{H^1}^2 + \|z\|_{H^1}^{\frac{4}{d}}\right)\frac{v_k}{2}.$$

Since $\int_{\mathbb{R}^d} \left[\nabla Q_{\omega_k} {}^t\nabla Q_{\omega_k}\right]\, dx$ is a positive definite symmetric matrix (by Proposition 2.30), it is invertible, so that inequality (2.128) holds for $|b_k'(t)|$. Then we conclude that the same inequality is true for $|a_k'(t)|$ by (3.99). □

2.3.3 End of the proof

We now conclude the proof of our uniqueness result, that is Theorem 2.3′ by gathering (2.94) and the different controls obtained in subsection 2.3.2.

Let us begin with a control of $\|z\|_{H^1}$ in terms of $\|\tilde{z}\|_{H^1}$ which relies on the integrability of $t \mapsto \|z(t)\|_{H^1} + \|z(t)\|_{H^1}^{\frac{4}{d}-1}$ in the neighborhood of $+\infty$ (provided N is chosen sufficiently large); once more, we observe here that the condition $d \le 3$ is important.

Proposition 2.23. *For t large enough,*

$$\|z(t)\|_{H^1} \le C\left(\sup_{s\ge t}\|\tilde{z}(s)\|_{H^1} + \int_t^{+\infty}\|\tilde{z}(u)\|_{H^1}\, du\right). \tag{2.131}$$

Proof. Recall that we have already seen (2.86):

$$\|z\|_{H^1} \le C\left(\|\tilde{z}\|_{H^1} + \sum_{k=1}^{K}\left(|a_k| + |b_k|\right)\right)$$

Therefore, using the control of the modulation parameters obtained before, that is (2.128),

$$\|z(t)\|_{H^1} \le C\|\tilde{z}(t)\|_{H^1} + C\left(\int_t^{+\infty}\|\tilde{z}(s)\|_{H^1}\, ds + \int_t^{+\infty}\left(\|z(s)\|_{H^1}^2 + \|z(s)\|_{H^1}^{\frac{4}{d}}\right) ds + \int_t^{+\infty} e^{-\gamma s}\|z(s)\|_{H^1}\, ds\right). \tag{2.132}$$

Since $t \mapsto \|z(t)\|_{H^1} + \|z(t)\|_{H^1}^{\frac{4}{d}-1}$ is integrable in the neighborhood of $+\infty$, we have for t large enough:

$$\int_t^{+\infty}\left(\|z(s)\|_{H^1}^2 + \|z(s)\|_{H^1}^{\frac{4}{d}}\right) ds \le \left(\int_t^{+\infty}\left(\|z(s)\|_{H^1} + \|z(s)\|_{H^1}^{\frac{4}{d}-1}\right) ds\right)\sup_{s\ge t}\|z(s)\|_{H^1}. \tag{2.133}$$

Similarly we have

$$\int_t^{+\infty} e^{-\gamma s}\|z(s)\|_{H^1}\, ds \le \frac{e^{-\gamma t}}{\gamma}\sup_{s\ge t}\|z(s)\|_{H^1}. \tag{2.134}$$

It follows from (2.132), (2.133), and (2.134) that for t large enough,

$$\sup_{s\geq t}\|z(s)\|_{H^1} \leq C\left(\sup_{s\geq t}\|\tilde{z}(s)\|_{H^1} + \int_t^{+\infty}\|\tilde{z}(s)\|_{H^1}\,ds\right) + C\left(\int_t^{+\infty}\left(\|z(s)\|_{H^1} + \|z(s)\|_{H^1}^{\frac{4}{d}-1}\right)\,ds + e^{-\gamma t}\right)\sup_{s\geq t}\|z(s)\|_{H^1}. \tag{2.135}$$

Hence, for t large enough,

$$\sup_{s\geq t}\|z(s)\|_{H^1} \leq C\left(\sup_{s\geq t}\|\tilde{z}(s)\|_{H^1} + \int_t^{+\infty}\|\tilde{z}(u)\|_{H^1}\,du\right), \tag{2.136}$$

which ends the proof of Proposition 2.23. □

Now, we deduce the following

Lemma 2.24. *There exists $T > 0$ such that for all $t \geq T$, $\tilde{z}(t) = 0$.*

Proof. By means of (2.94), (2.100), and (2.127), we can write for t large enough

$$\begin{aligned}\|\tilde{z}(t)\|_{H^1}^2 &\leq C\int_t^{+\infty}\left(\frac{1}{s}\|\tilde{z}(s)\|_{H^1}^2 + e^{-\gamma s}\|z(s)\|_{H^1}\|\tilde{z}(s)\|_{H^1} + \|z(s)\|_{H^1}^2\|\tilde{z}(s)\|_{H^1}\right)\,ds \\ &\leq C\left[\int_t^{+\infty}\left(\frac{1}{s}\|\tilde{z}(s)\|_{H^1} + e^{-\gamma s}\|z(s)\|_{H^1} + \|z(s)\|_{H^1}^2\right)\,ds\right]\sup_{s\geq t}\|\tilde{z}(s)\|_{H^1}.\end{aligned} \tag{2.137}$$

We deduce from the preceding line that for t large enough

$$\|\tilde{z}(t)\|_{H^1} \leq C\int_t^{+\infty}\left(\frac{1}{s}\|\tilde{z}(s)\|_{H^1} + e^{-\gamma s}\|z(s)\|_{H^1} + \|z(s)\|_{H^1}^2\right)\,ds. \tag{2.138}$$

Using (2.133), (2.134), and (2.136), this leads to the fact that for t large enough

$$\|\tilde{z}(t)\|_{H^1} \leq C\int_t^{+\infty}\frac{1}{s}\|\tilde{z}(s)\|_{H^1}\,ds + C\left(e^{-\gamma t} + \int_t^{+\infty}\|z(s)\|_{H^1}\,ds\right)\left(\sup_{s\geq t}\|\tilde{z}(s)\|_{H^1} + \int_t^{+\infty}\|\tilde{z}(u)\|_{H^1}\,du\right).$$

Thus, for large values of t,

$$\|\tilde{z}(t)\|_{H^1} \leq C\left[\int_t^{+\infty}\frac{1}{s}\|\tilde{z}(s)\|_{H^1}\,ds + \left(e^{-\gamma t} + \int_t^{+\infty}\|z(s)\|_{H^1}\,ds\right)\left(\int_t^{+\infty}\|\tilde{z}(u)\|_{H^1}\,du\right)\right]. \tag{2.139}$$

Since by assumption $\|z(t)\|_{H^1} = \mathrm{O}\left(\frac{1}{t^\alpha}\right)$ with $\alpha > 2$ and since $e^{-\gamma t} \leq \frac{1}{t^{\alpha-1}}$ for t large enough, there exist $\tilde{C} \geq 0$ and $T \geq 1$ such that for all $t \geq T$,

$$\|\tilde{z}(t)\|_{H^1} \leq \tilde{C}\left[\int_t^{+\infty}\frac{1}{s}\|\tilde{z}(s)\|_{H^1}\,ds + \frac{1}{t^{\alpha-1}}\left(\int_t^{+\infty}\|\tilde{z}(u)\|_{H^1}\,du\right)\right]. \tag{2.140}$$

Remark 2.9. Note that in (2.140), $\tilde{C}$ seems to depend on z (or equivalently on u) but in fact it does not (even if it means changing T which does actually depend on z). Indeed, $\tilde{C}$ depends only on the constants appearing in (2.74), (2.86), (2.87), (2.94) (linked with the parameters used to define the solitons R_k), on the constants appearing in (2.96), (2.100), (2.127), (2.128) (linked with f, with the parameters used to define the solitons, and with $\|z(t)\|_{H^1}$ which can be chosen less or equal to 1 provided t is sufficiently large, depending on z), and also on universal constants which enable us to pass from (2.135) to (2.136), from (2.137) to (2.138), and from (2.139) to (2.140) (on condition that t is once more sufficiently large, which depends on z). Thus one should read the following assertion: there exists $\tilde{C} > 0$ such that for all z satisfying (2.76) and $\|z(t)\|_{H^1} = \mathrm{O}\left(\frac{1}{t^\alpha}\right)$, there exists $T(z) > 0$ such that for all $t \geq T(z)$, (2.140) holds.

Now take N in (2.77) such that $N > 4\tilde{C}+1$ (in this way, N does not depend on z, as emphasized in Remark 2.9). Even if it means taking a larger T, we can assume

$$\exists\, c \geq 0,\ \forall\, t \geq T, \quad \|\tilde{z}(t)\|_{H^1} \leq \frac{c}{t^N}.$$

Then $A := \sup_{t\geq T}\{t^N \|\tilde{z}(t)\|_{H^1}\}$ is well defined. Let us pick up $\tilde{T} \geq T$ such that $\tilde{T}^N \|\tilde{z}(\tilde{T})\|_{H^1} \geq \frac{A}{2}$.

Now, replacing t by $\tilde{T}$ in (2.140), we obtain

$$\frac{A}{2\tilde{T}^N} \leq \tilde{C}A\left(\frac{1}{N\tilde{T}^N} + \frac{1}{(N-1)\tilde{T}^{N+\alpha-2}}\right) \leq \frac{2\tilde{C}A}{(N-1)\tilde{T}^N}. \tag{2.141}$$

Supposing $A \neq 0$ would lead to a contradiction because of the choice of N.
Consequently

$$\forall\, t \geq T, \qquad \|\tilde{z}(t)\|_{H^1} = 0.$$

□

We deduce from Proposition 2.23 and Lemma 2.24 that

$$\forall\, t \geq T, \qquad \|z(t)\|_{H^1} = 0.$$

The local well-posedness in $H^1(\mathbb{R}^d)$ of (NLS) implies then $u = \varphi$. Hence Theorem 2.3′ is proved.

2.3.4 Uniqueness result for the critical pure-power case

In this paragraph, let $d \geq 1$ and $f : r \mapsto r^{\frac{2}{d}}$. Our proof of uniqueness in the class of multi-solitons u such that $\|u(t) - R(t)\|_{H^1} \underset{t\to+\infty}{=} \mathrm{O}\left(\frac{1}{t^N}\right)$ (for some $N \in \mathbb{N}^*$ sufficiently large to be determined later) and in the L^2-critical case consists in exploiting the same ideas as for the subcritical case. Nevertheless it is this time based on Proposition 2.25, stated below and proved in Appendix.

Proposition 2.25. *Assume that $f(r) = r^{\frac{2}{d}}$ and let $\omega > 0$. There exists $\mu > 0$ such that for all $w = w_1 + iw_2 \in H^1(\mathbb{R}, \mathbb{C})$,*

$$\begin{aligned} H(w) \geq \mu\|w\|_{H^1}^2 - \frac{1}{\mu}\left(\int_{\mathbb{R}^d} w_1 Q_\omega dx\right)^2 \\ - \frac{1}{\mu}\left(\sum_{i=1}^d \left(\int_{\mathbb{R}^d} w_1 \partial_{x_i} Q_\omega\, dx\right)^2 + \left(\int_{\mathbb{R}^d} w_1\, (x\cdot\nabla Q_\omega)^2\, dx\right)^2 + \left(\int_{\mathbb{R}^d} w_2 Q_\omega\, dx\right)^2\right). \end{aligned} \tag{2.142}$$

In order not to be too redundant, we only explicit the main modifications of the proof given for the stable case.

Change of variable

We still consider φ, u, and z as defined at the beginning of Section 2.3. In order to apply Proposition 2.25 (which states a coercivity property available in the critical case), one has to take into account a third family of directions indexed by $k = 1, \dots, K$. More precisely, let us introduce $y_k(t,x) := x - v_k t - x_k^0 \in \mathbb{R}^d$, for all $k = 1, \dots, K$, and

$$\tilde{z}(t,x) := z(t,x) + \sum_{k=1}^{K} \{ia_k(t)R_k(t,x) + b_k(t) \cdot \nabla R_k(t,x)\} + \sum_{k=1}^{K} c_k(t) \left(\frac{d}{2}R_k + y_k \cdot \nabla R_k - \frac{i}{2}v_k \cdot y_k R_k\right)(t,x),$$

where a_k, b_k, and c_k are well defined on $[T_1, +\infty)$ (even if it means taking a larger T_1) with values respectively in $\mathbb{R}$, $\mathbb{R}^d$, and $\mathbb{R}$ such that

$$\forall\, t \geq T_1, \quad \begin{cases} \operatorname{Im} \displaystyle\int_{\mathbb{R}^d} \tilde{z}(t)\overline{R_k(t)}\, dx = 0 \\ \operatorname{Re} \displaystyle\int_{\mathbb{R}^d} \tilde{z}(t)\overline{\nabla R_k(t)}\, dx = 0 \\ \operatorname{Re} \displaystyle\int_{\mathbb{R}^d} \tilde{z}(t)\overline{\left(\frac{d}{2}R_k + y_k \cdot \nabla R_k - \frac{i}{2}v_k \cdot y_k R_k\right)(t)}\, dx = 0. \end{cases} \tag{2.143}$$

As for the stable case, we can prove that $a_k(t)$, $b_k(t)$, and $c_k(t)$, $k = 1, \dots, K$, are uniquely determined by the preceding orthogonality conditions. This time, we have to show indeed that the following block matrix is invertible:

$$\tilde{M}(t) := \begin{bmatrix} A_{0,0}(t) & B_{1,1}(t) & \cdots & B_{1,d}(t) & Z_0(t) \\ {}^tB_{1,1}(t) & A_{1,1}(t) & \cdots & A_{1,d}(t) & Z_1(t) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ {}^tB_{1,d}(t) & A_{d,1}(t) & \cdots & A_{d,d}(t) & Z_d(t) \\ {}^tZ_0(t) & {}^tZ_1(t) & \cdots & {}^tZ_d(t) & W(t) \end{bmatrix},$$

where $A_{0,0}$, $A_{i,j}$, $B_{1,j}$ are defined in paragraph 2.3.1, $Z_i(t)$ ($i = 0, \dots, d$) has entries zero on its diagonal and $\mathrm{O}(e^{-\gamma t})$ elsewhere, and $W(t)$ possesses the coefficients $\int_{\mathbb{R}^d} \left(x \cdot \nabla Q_{\omega_k}\right)^2\, dx$, $k = 1, \dots, K$ on its diagonal and $\mathrm{O}(e^{-\gamma t})$ elsewhere.
For the sake of completeness, let us justify how to determine the coefficients of $Z_i(t)$ ($i = 1, \dots, d$) which are the less obvious ones to compute. By the orthogonality condition $\operatorname{Re} \int_{\mathbb{R}^d} \tilde{z}(t)\nabla\overline{R_k}(t)\, dx = 0$, the coefficient $Z_i(k,l)$ of Z_i located at line k and column l is equal to

$$Z_i(k,l) = \frac{d}{2}\operatorname{Re} \int_{\mathbb{R}^d} R_k \partial_{x_i}\overline{R_l}\, dx + \operatorname{Re} \int_{\mathbb{R}^d} y_k \cdot \nabla R_k \partial_{x_i}\overline{R_l}\, dx + \frac{1}{2}v_k \cdot \operatorname{Im} \int_{\mathbb{R}^d} y_k R_k \partial_{x_i}\overline{R_l}\, dx.$$

Thus for $k \neq l$, we have $Z_i(k,l) = \mathrm{O}(e^{-\gamma t})$ by (2.24), and for $k = l$, we obtain:

$$Z_i(k,l) = \mathrm{Re} \int_{\mathbb{R}^d} y_k \cdot \left(\nabla R_k - \frac{i}{2} v_k R_k \right) \partial_{x_i} \overline{R_k}\, dx = \int_{\mathbb{R}^d} x \cdot \nabla Q_{\omega_k} \partial_{x_i} Q_{\omega_k}$$
$$= \sum_{j=1}^{d} \int_{\mathbb{R}^{d-1}} \left(\int_{\mathbb{R}} x_j \partial_{x_j} Q_{\omega_k} \partial_{x_i} Q_{\omega_k}\, dx_i \right) dx_1 \dots \hat{dx_i} \dots dx_d = 0,$$

since for all $j \in \{1, \dots, d\}$, $x_i \mapsto x_j \partial_{x_j} Q_{\omega_k}(x_1, \dots, x_d) \partial_{x_i} Q_{\omega_k}(x_1, \dots, x_d)$ is an odd integrable function on $\mathbb{R}$ in view of the fact that Q_{ω_k} is radial.

Hence, we obtain that

$$\det \tilde{M}(t) = \det M(t) \prod_{k=1}^{K} \int_{\mathbb{R}^d} \left(x \cdot \nabla Q_{\omega_k} \right)^2\, dx + \mathrm{O}(e^{-\gamma t})$$

is strictly positive for t large enough (see (2.167) in Appendix).

Moreover, for all $t \geq T_1$,

$$|a_k(t)|, |b_k(t)|, |c_k(t)| \leq \|z(t)\|_{L^2}, \tag{2.144}$$

and

$$|a_k'(t)|, |b_k'(t)|, |c_k'(t)| \leq \|z(t)\|_{H^1}. \tag{2.145}$$

Remark 2.10. The consideration of $\tilde{z}$ turns out to be appropriate judging by the properties stated in Lemma 2.27. Besides let us note that the particular non-linearity satisfies the ODE $dxf'(x) = 2f(x)$ in the L^2-critical case; this will be truly useful to control the third family of directions associated with the coefficients c_k.

First of all, let us begin with the useful computation of the derivative of $\tilde{z}$ with respect to the time variable.

Lemma 2.26. *We have*

$$\partial_t \tilde{z} = i\Big(\Delta \tilde{z} + f(|\varphi|^2)\tilde{z} + \big(f(|\tilde{z} + \varphi|^2) - f(|\varphi|^2) \big)\varphi \Big) + \sum_{k=1}^{K} \left\{ i a_k' R_k + b_k' \cdot \nabla R_k \right\}$$
$$- 2i \sum_{k=1}^{K} c_k \omega_k R_k + \sum_{k=1}^{K} c_k' \left(\frac{d}{2} R_k + y_k \cdot \nabla R_k - \frac{i}{2} v_k \cdot y_k R_k(t,x) \right) + \epsilon_1,$$

where ϵ_1 *is a function of t and x such that*

$$\int_{\mathbb{R}^d} |\epsilon_1| \big(|\tilde{z}| + |\nabla \tilde{z}| + |\Delta \tilde{z}| \big)\, dx \leq C\big(e^{-\gamma t} \|z\|_{H^1} \|\tilde{z}\|_{H^1} + \|z\|_{H^1}^2 \|\tilde{z}\|_{H^1} \big).$$

Proof. Note that $\partial_t \tilde{z}$ decomposes like

$$\begin{aligned}\partial_t \tilde{z} = \partial_t z &+ \sum_{k=1}^{K} \left\{ a_k'(\cdots) + b_k'(\cdots) + a_k(\cdots) + b_k(\cdots) \right\} \\ &+ \sum_{k=1}^{K} c_k' \left(\frac{d}{2} R_k + y_k \cdot \nabla R_k - \frac{i}{2} v_k \cdot y_k R_k \right) \\ &+ \sum_{k=1}^{K} c_k \Bigg(i \frac{d}{2} (\Delta R_k + f(|R_k|^2)) R_k) - v_k \cdot \nabla R_k + \frac{i}{2} |v_k|^2 R_k + i y_k \cdot \nabla \big(\Delta R_k + f(|R_k|^2) R_k \big) \\ &\qquad + \frac{1}{2} v_k \cdot y_k \big(\Delta R_k + f(|R_k|^2) R_k \big) \Bigg).\end{aligned} \tag{2.146}$$

Now, we want to express $\partial_t z$ given by (2.76) in terms of $\tilde{z}$, as already made in Lemma 2.19. For this, let us observe that:

$$\Delta \big(y_k \cdot \nabla R_k \big) = y_k \cdot \nabla (\Delta R_k) + 2 \Delta R_k, \tag{2.147}$$

$$\Delta \left(-\frac{i}{2} v_k \cdot y_k R_k \right) = -\frac{i}{2} v_k \cdot y_k \Delta R_k - i v_k \cdot \nabla R_k, \tag{2.148}$$

$$\begin{aligned} f(|z+\varphi|^2) z = {} & \big(f(|z+\varphi|^2) - f(|\varphi|^2) \big) z + f(|\varphi|^2) \tilde{z} \\ & - f(|\varphi|^2) \sum_k \left\{ a_k(\cdots) + b_k(\cdots) + c_k \left(\frac{d}{2} R_k + y_k \cdot \nabla R_k - \frac{i}{2} v_k \cdot y_k R_k(t,x) \right) \right\}, \end{aligned} \tag{2.149}$$

and

$$\begin{aligned} \big(f(|z+\varphi|^2) - f(|\varphi|^2) \big) \varphi &= \big(f(|\tilde{z}+\varphi|^2) - f(|\varphi|^2) \big) \varphi + 2 \mathrm{Re} \big((z - \tilde{z}) \overline{\varphi} \big) f'(|\varphi|^2) \varphi + \epsilon_1 \\ &= \big(f(|\tilde{z}+\varphi|^2) - f(|\varphi|^2) \big) \varphi \\ &\quad - 2 \sum_{k=1}^{K} \mathrm{Re} \left[\left(b_k(\cdots) + c_k \left(\frac{d}{2} R_k + y_k \cdot \nabla R_k \right) \right) \overline{R_k} \right] f'(|R_k|^2) R_k + \epsilon_1. \end{aligned} \tag{2.150}$$

Inserting each equality (2.147), (2.148), (2.149), and (2.150) in (2.146) leads to

$$\begin{aligned} \partial_t \tilde{z} = i \left(\Delta \tilde{z} + f(|\varphi|^2) \tilde{z} + \big(f(|\tilde{z}+\varphi|^2) - f(|\varphi|^2) \big) \varphi \right) &+ \sum_{k=1}^{K} \left\{ i a_k' R_k + b_k' \cdot \nabla R_k \right\} \\ + \sum_{k=1}^{K} c_k' \left(\frac{d}{2} R_k + y_k \cdot \nabla R_k - \frac{i}{2} v_k \cdot y_k R_k(t,x) \right) & \\ + \sum_{k=1}^{K} c_k \left(-2 v_k \cdot \nabla R_k + \frac{i}{2} |v_k|^2 R_k - 2 i \Delta R_k - i d |R_k|^2 f'(|R_k|^2) R_k \right) &+ \epsilon_1. \end{aligned}$$

Finally, we conclude by means of

$$d \, |R_k|^2 f'(|R_k|^2) = 2 f(|R_k|^2) \tag{2.151}$$

(which indeed holds in the L^2-critical case as mentioned in Remark 2.10) and the two possibilities given in (2.112) to write $\partial_t R_k$. □

Control of $\tilde{H}$ and of the modulation parameters

Take again $\tilde{H}$ as defined at the end of paragraph 2.3.1 and consider still $Main(t)$ as in Proposition 2.17. Then we can state

Lemma 2.27 (Control of the derivative of the Weinstein functional)**.** *The following assertion holds true:*

$$\frac{d\tilde{H}}{dt}(t) = Main(t) - 4\sum_{k=1}^{K} c_k'(t)\left(\omega_k - \frac{|v_k|^2}{4}\right)\text{Re}\int_{\mathbb{R}^d} \tilde{z}\overline{R_k}(t)\,dx$$
$$+ \text{O}\left(\left(e^{-\gamma t} + \|z(t)\|_{H^1}\right)\|\tilde{z}(t)\|_{H^1}\|z(t)\|_{H^1}\right).$$

Proof. Take again the proof of Proposition 2.17. Concerning the expression of the derivative of $\tilde{H}$, observe that everything is kept unchanged in the present context except that we have to take care of the additional terms involving the parameters $c_k(t)$ and $c_k'(t)$, for all $k = 1,\dots,K$.

Let us define the $\mathbb{C}$-linear endomorphism $\mathcal{L}_k$ of $H^1(\mathbb{R}^d)$ by

$$\mathcal{L}_k(v) := -\Delta v - f(|R_k|^2)v + \left(\omega_k + \frac{|v_k|^2}{4}\right)v + iv_k\cdot\nabla v.$$

Observe that $\mathcal{L}_k(R_k) = 0$ and for all $v, w \in H^1(\mathbb{R}^d)$,

$$\text{Re}\int_{\mathbb{R}^d} v\overline{\mathcal{L}_k(w)}\phi_k\,dx = \text{Re}\int_{\mathbb{R}^d}\overline{w}\mathcal{L}_k(v)\phi_k\,dx - 2\text{Re}\int_{\mathbb{R}^d}\overline{w}\nabla v\cdot\nabla\phi_k\,dx + \text{Re}\,i\int_{\mathbb{R}^d} v\overline{w}v_k\cdot\nabla\phi_k\,dx, \tag{2.152}$$

Using (2.90), (2.92), and (2.152), we deduce that for all $v \in H^1(\mathbb{R}^d)$:

$$\begin{aligned}\text{Re}\int_{\mathbb{R}^d} R_k\overline{\mathcal{L}_k(v)}\,dx &= \text{Re}\int_{\mathbb{R}^d} R_k\overline{\mathcal{L}_k(v)}\phi_k\,dx + \text{O}\left(e^{-\gamma t}\|v\|_{H^1}\right)\\ &= \text{Re}\int_{\mathbb{R}^d} v\overline{\mathcal{L}_k(R_k)}\phi_k\,dx + \text{O}\left(e^{-\gamma t}\|v\|_{H^1}\right) = \text{O}\left(e^{-\gamma t}\|v\|_{H^1}\right).\end{aligned} \tag{2.153}$$

The term associated with $c_k(t)$ in the expression of $\frac{d}{dt}\tilde{H}$ writes

$$2\,c_k(t)\text{Re}\int_{\mathbb{R}^d}(-2i\omega_k R_k)\overline{\mathcal{L}_k(\tilde{z})}\,dx.$$

By (2.153), it is thus bounded by $Ce^{-\gamma t}\|\tilde{z}\|_{H^1}\|z\|_{H^1}$.

It remains us to obtain the term associated with c_k' in $\frac{d}{dt}\tilde{H}$. This term corresponds to $\mathcal{I}_{1,k} - \mathcal{I}_{2,k}$, where

$$\mathcal{I}_{1,k} = 2\,\text{Re}\int_{\mathbb{R}^d}\left(\frac{d}{2}R_k + y_k\cdot\nabla R_k - \frac{i}{2}v_k\cdot y_k R_k(t,x)\right)\overline{\mathcal{L}_k(\tilde{z})}\phi_k\,dx$$

and

$$\mathcal{I}_{2,k} = 2\,\text{Re}\int_{\mathbb{R}^d}\left(\frac{d}{2}R_k + y_k\cdot\nabla R_k\right)\overline{\varphi}\big(f(|\tilde{z}+\varphi|^2) - f(|\varphi|^2)\big)\,dx.$$

Let us concentrate first on $\mathcal{I}_{1,k}$. By (2.152),

$$\mathcal{I}_{1,k} = 2\,\mathrm{Re}\int_{\mathbb{R}^d} \mathcal{L}_k\left(\frac{d}{2}R_k + y_k\cdot\nabla R_k - \frac{i}{2}v_k\cdot y_k R_k(t,x)\right)\bar{\tilde{z}}\,dx + \mathrm{O}\left(e^{-\gamma t}\|\tilde{z}\|_{H^1}\right).$$

Moreover,

$$\begin{aligned}
&\mathrm{Re}\int_{\mathbb{R}^d}\bar{\tilde{z}}\mathcal{L}_k\left(\frac{d}{2}R_k\right)\,dx = 0;\\
&\mathrm{Re}\int_{\mathbb{R}^d}\bar{\tilde{z}}\mathcal{L}_k(y_k\cdot\nabla R_k)\,dx\\
&\quad= \mathrm{Re}\int_{\mathbb{R}^d}\bar{\tilde{z}}\big(-y_k\cdot\nabla(\Delta R_k) - 2\Delta R_k\big)\,dx - \mathrm{Re}\int_{\mathbb{R}^d}\bar{\tilde{z}}f(|R_k|^2)y_k\cdot\nabla R_k\,dx\\
&\qquad+\mathrm{Re}\int_{\mathbb{R}^d} i\bar{\tilde{z}}v_k\cdot\nabla(y_k\cdot\nabla R_k)\,dx + \mathrm{Re}\int_{\mathbb{R}^d}\bar{\tilde{z}}\left(\omega_k+\frac{|v_k|^2}{4}\right)y_k\cdot\nabla R_k\,dx\\
&\quad= \mathrm{Re}\int_{\mathbb{R}^d}\left(\nabla\bar{\tilde{z}}\cdot y_k + d\bar{\tilde{z}}\right)\Delta R_k\,dx - 2\,\mathrm{Re}\int_{\mathbb{R}^d}\bar{v}\Delta R_k\,dx\\
&\qquad+\mathrm{Re}\int_{\mathbb{R}^d}\left(\nabla\bar{\tilde{z}}\cdot y_k + d\bar{\tilde{z}}\right)f(|R_k|^2)R_k\,dx + \mathrm{Re}\int_{\mathbb{R}^d}\bar{\tilde{z}}y_k\cdot\nabla(f(|R_k|^2))R_k\,dx\\
&\qquad-\mathrm{Re}\int_{\mathbb{R}^d} i\left(\nabla\bar{\tilde{z}}\cdot y_k\right)\left(v_k\cdot\nabla R_k\right)\,dx - (d-1)\mathrm{Re}\int_{\mathbb{R}^d} i\bar{\tilde{z}}v_k\cdot\nabla R_k\,dx\\
&\qquad-\mathrm{Re}\int_{\mathbb{R}^d}\left(\nabla\bar{\tilde{z}}\cdot y_k + d\bar{\tilde{z}}\right)\left(\omega_k+\frac{|v_k|^2}{4}\right)R_k\,dx + \mathrm{O}\left(e^{-\gamma t}\|\tilde{z}\|_{L^2}\right)\\
&\quad= -2\,\mathrm{Re}\int_{\mathbb{R}^d}\bar{\tilde{z}}\Delta R_k\,dx + \mathrm{Re}\int_{\mathbb{R}^d} i\bar{\tilde{z}}v_k\cdot\nabla R_k\,dx\\
&\qquad+2\,\mathrm{Re}\int_{\mathbb{R}^d}\bar{\tilde{z}}f'(|R_k|^2)y_k\cdot\mathrm{Re}\left(\overline{R_k}\nabla R_k\right)R_k\,dx + \mathrm{O}(e^{-\gamma t}\|\tilde{z}\|_{L^2});\\
&\mathrm{Re}\int_{\mathbb{R}^d}\bar{\tilde{z}}\mathcal{L}_k\left(\frac{i}{2}v_k\cdot y_k R_k\right)\,dx\\
&\quad= \mathrm{Re}\int_{\mathbb{R}^d}\bar{\tilde{z}}\left(-\frac{i}{2}(v_k\cdot y_k)\Delta R_k - iv_k\cdot\nabla R_k\right)\,dx + \mathrm{Re}\int_{\mathbb{R}^d}\bar{\tilde{z}}\left(\omega_k+\frac{|v_k|^2}{4}\right)\frac{i}{2}v_k\cdot y_k R_k\,dx\\
&\qquad-\frac{1}{2}\mathrm{Re}\int_{\mathbb{R}^d}\bar{\tilde{z}}v_k\cdot\nabla\left(v_k\cdot y_k R_k\right)\,dx - \mathrm{Re}\int_{\mathbb{R}^d}\frac{i}{2}\bar{\tilde{z}}f(|R_k|^2)v_k\cdot y_k R_k\,dx + \mathrm{O}\left(e^{-\gamma t}\|\tilde{z}\|_{L^2}\right)\\
&\quad= -\mathrm{Re}\int_{\mathbb{R}^d} i\bar{\tilde{z}}v_k\cdot\nabla R_k\,dx - \frac{|v_k|^2}{2}\mathrm{Re}\int_{\mathbb{R}^d}\bar{\tilde{z}}R_k\,dx + \mathrm{O}\left(e^{-\gamma t}\|\tilde{z}\|_{L^2}\right).
\end{aligned}$$

Note that to establish the three preceding equalities, we have used once again $\mathcal{L}_k(R_k) = 0$. Thus, gathering the preceding calculations, we infer

$$\begin{aligned}
c_k'\mathcal{I}_{1,k} &= 2\,c_k'\mathrm{Re}\int_{\mathbb{R}^d}\bar{\tilde{z}}R_k y_k\cdot\nabla\big(f(|R_k|^2)\big)\,dx_k - 4\,c_k'\mathrm{Re}\int_{\mathbb{R}^d}\bar{\tilde{z}}\left(\Delta R_k + iv_k\cdot\nabla R_k - \frac{|v_k|^2}{4}R_k\right)\,dx\\
&\quad+\mathrm{O}\left(e^{-\gamma t}\|\tilde{z}\|_{L^2}\|z\|_{H^1}\right).
\end{aligned} \tag{2.154}$$

Now, let us focus on the second integral $\mathcal{I}_{2,k}$. On the one hand, by means of a Taylor expansion, by (2.24), (2.75), and (2.151), we obtain

$$\begin{aligned} d\int_{\mathbb{R}^d} |R_k|^2\big(f(|\tilde z+\varphi|^2)-f(|\varphi|^2)\big)\,dx \\ &= d\int_{\mathbb{R}^d} |R_k|^2\times 2\mathrm{Re}(\tilde z R_k) f'(|R_k|^2)\,dx + \mathrm{O}\big(e^{-\gamma t}\|\tilde z\|_{L^2}+\|\tilde z\|_{L^2}^2\big) \\ &= 4\int_{\mathbb{R}^d} f(|R_k|^2)\mathrm{Re}(\tilde z R_k)\,dx + \mathrm{O}\big(e^{-\gamma t}\|\tilde z\|_{L^2}+\|\tilde z\|_{L^2}^2\big). \end{aligned} \tag{2.155}$$

On the other, we observe that

$$\begin{aligned} 2\,\mathrm{Re}\int_{\mathbb{R}^d} y_k\cdot\nabla R_k\overline{R_k}\big(f(|\tilde z+\varphi|^2)-f(|\varphi|^2)\big)\,dx \\ = 4\,\mathrm{Re}\int_{\mathbb{R}^d} y_k\cdot\nabla R_k\overline{R_k}\mathrm{Re}\left(\tilde z\overline{R_k}\right) f'(|R_k|^2)\,dx + \mathrm{O}\big(e^{-\gamma t}\|\tilde z\|_{L^2}+\|\tilde z\|_{L^2}^2\big). \end{aligned} \tag{2.156}$$

Thus, we deduce from (2.155), (2.156), and (**??**) that

$$\begin{aligned} c_k'\mathcal{I}_{2,k} = c_k'\mathrm{Re}\int_{\mathbb{R}^d} \overline{\tilde z}\left(\Delta R_k + i v_k\cdot\nabla R_k - \frac{|v_k|^2}{4}R_k\right)\,dx \\ + 2\,c_k'\mathrm{Re}\int_{\mathbb{R}^d} \overline{\tilde z}R_k y_k\cdot\nabla\left(f(|R_k|^2)\right)\,dx + \mathrm{O}\big(e^{-\gamma t}\|\tilde z\|_{L^2}+\|\tilde z\|_{L^2}^2\big). \end{aligned} \tag{2.157}$$

From (2.154) and (2.157), we conclude that the term associated with c_k' in $\frac{d}{dt}\tilde H$ is equal to

$$-4\,c_k'\mathrm{Re}\int_{\mathbb{R}^d} \overline{\tilde z}\left(\Delta R_k + i v_k\cdot\nabla R_k + f(|R_k|^2)R_k - \frac{|v_k|^2}{4}R_k\right)\,dx + \mathrm{O}(e^{-\gamma t}\|\tilde z\|_{L^2}\|z\|_{H^1}),$$

that is to

$$4\,c_k'\left(\omega_k - \frac{|v_k|^2}{4}\right)\mathrm{Re}\int_{\mathbb{R}^d}\overline{\tilde z}R_k\,dx + \mathrm{O}(e^{-\gamma t}\|\tilde z\|_{L^2}\|z\|_{H^1}).$$

This finishes the proof of Lemma 2.27. □

Lemma 2.28 (Control of the modulation parameters)**.** *We have for all $k=1,\dots,K$:*

$$\forall\, t\geq T_1,\quad |a_k'(t)|+|b_k'(t)|+|c_k'(t)| \leq C\big(e^{-\gamma t}\|z\|_{H^1}+\|z\|_{H^1}^2+\|\tilde z\|_{L^2}\big).$$

Proof. This lemma follows from the preliminary computations

$$\begin{aligned} \Delta\tilde z =& \Delta z + \sum_{j=1}^K\left\{ i a_j\Delta R_k + b_j\cdot\nabla(\Delta R_j) + c_j\left(\left(\frac{d}{2}+2\right)\Delta R_j + y_j\cdot\nabla(\Delta R_j)\right)\right\} \\ &- \sum_{j=1}^K\left\{\frac{i}{2}v_j\cdot y_j\Delta R_j + i v_j\cdot\nabla R_j\right\} \\ \partial_t\tilde z =& \partial_t z + \sum_{j=1}^K\left\{ i a_j' R_j + b_j'\cdot\nabla R_j + c_j'\left(\frac{d}{2}R_j + y_j\cdot\nabla R_j - \frac{i}{2}v_j\cdot y_j R_j\right)\right\} \end{aligned}$$

$$+\sum_{j=1}^{K}\left\{ia_j\partial_t R_j + b_j\cdot\nabla(\partial_t R_j)\right\}$$
$$+\sum_{j=1}^{K}c_j\left(\frac{d}{2}\partial_t R_j + y_j\cdot\nabla(\partial_t R_j) - \frac{i}{2}v_j\cdot y_j\partial_t R_j - v_j\cdot\nabla R_j + \frac{i}{2}|v_j|^2 R_j\right),$$

from (3.99), (3.100) (which still take the same form in the present context with the consideration of a third direction), and from the derivation with respect to t of the third family of orthogonal conditions

$$\mathrm{Re}\int_{\mathbb{R}^d}\tilde{z}\left(\overline{\frac{d}{2}R_k + y_k\cdot\nabla R_k - \frac{i}{2}v_k\cdot y_k R_k}\right)\,dx = 0,$$

which yields

$$0 = c_k'\left(\int_{\mathbb{R}^d}\left(x\cdot\nabla Q_{\omega_k}\right)^2\,dx - \frac{d^2}{4}\int_{\mathbb{R}^d}Q_{\omega_k}^2\,dx\right) + \mathrm{O}\big(\|z\|_{H^1}^2 + e^{-\gamma t}\|z\|_{H^1} + \|\tilde{z}\|_{H^1}\big). \quad (2.158)$$

We finish the proof of Lemma 2.28 with (2.167) in Appendix. □

Now, to exploit Lemma 2.27 in order to perform an estimate of $\tilde{H}(t)$, we have to control the scalar products $\mathrm{Re}\int_{\mathbb{R}^d}\tilde{z}\overline{R_k}\,dx$, $k = 1,\dots,K$. In fact, we state and prove next the analogue of Lemma 2.20.

Lemma 2.29 (Control of the R_k directions)**.** *We have*

$$\left|\frac{d}{dt}\mathrm{Re}\int_{\mathbb{R}^d}\tilde{z}\overline{R_k}\,dx\right| \le C\left(e^{-\gamma t}\|z\|_{H^1} + \|z\|_{H^1}^2\right).$$

Proof. Note that (2.123) is still guaranteed here by observing moreover that

$$\begin{aligned}\sum_{j=1}^{K}c_j\mathrm{Re}\int_{\mathbb{R}^d}\left(\frac{d}{2}R_j + y_j\cdot\nabla R_j - \frac{i}{2}v_j R_j\right)\overline{R_k}\,dx &= \int_{\mathbb{R}^d}\frac{d}{2}|R_k|^2\,dx + \mathrm{Re}\int_{\mathbb{R}^d}y_k\cdot\nabla R_k\overline{R_k}\,dx + \epsilon_2(t)\\ &= \epsilon_2(t),\end{aligned}$$

where $\epsilon_2'(t) = \mathrm{O}\left(e^{-\gamma t}\|z(t)\|_{H^1}\right)$. Now, the rest of the proof of Lemma 2.20 is kept unchanged and thus we obtain Lemma 2.29. □

We deduce from Lemma 2.28 and from Lemma 2.29 that

$$\left|c_k'\mathrm{Re}\int_{\mathbb{R}^d}\tilde{z}\overline{R_k}\,dx\right| \le C\left(e^{-\gamma t}\|z\|_{H^1} + \|z\|_{H^1}^2 + \|\tilde{z}\|_{H^1}\right)\int_t^{+\infty}\left(e^{-\gamma s}\|z\|_{H^1} + \|z\|_{H^1}^2\right)\,ds. \quad (2.159)$$

Next, it follows from (2.159) and Lemma 2.27 that

$$\begin{aligned}&\tilde{H}(t)\\ &\le C\int_t^{+\infty}\left\{\frac{1}{s}\|\tilde{z}(s)\|_{H^1}^2 + \left(e^{-\gamma s} + \|z(s)\|_{H^1}\right)\|\tilde{z}(s)\|_{H^1}\|z(s)\|_{H^1}\right\}\,ds\\ &\quad + C\int_t^{+\infty}\left(e^{-\gamma s}\|z(s)\|_{H^1} + \|z(s)\|_{H^1}^2 + \|\tilde{z}(s)\|_{H^1}\right)\int_s^{+\infty}\left(e^{-\gamma u}\|z(u)\|_{H^1} + \|z(u)\|_{H^1}^2\right)\,du\,ds.\end{aligned} \quad (2.160)$$

Conclusion of the proof of uniqueness in the critical case

Proposition 2.25 allows us to obtain the coercivity estimate in Proposition 2.15) so that by (2.127) (which follows also from Lemma 2.29) and (2.160), we obtain

$$\begin{aligned}
&\|\tilde{z}(t)\|_{H^1}^2 \\
&\quad \leq C \int_t^{+\infty} \left\{ \frac{1}{s} \|\tilde{z}(s)\|_{H^1}^2 + (e^{-\gamma s} + \|z(s)\|_{H^1}) \|\tilde{z}(s)\|_{H^1} \|z(s)\|_{H^1} \right\} ds \\
&\quad + C \int_t^{+\infty} \left(e^{-\gamma s} \|z(s)\|_{H^1} + \|z(s)\|_{H^1}^2 + \|\tilde{z}(s)\|_{H^1} \right) \int_s^{+\infty} \left(e^{-\gamma u} \|z(u)\|_{H^1} + \|z(u)\|_{H^1}^2 \right) du\, ds.
\end{aligned}$$

On the other hand, Proposition 2.23 and in fact (2.133), (2.134), and (2.136) are still available here in the critical case; this is guaranteed by Lemma 2.28.

Thus, adapting the proof of Lemma 2.24, we deduce the following estimate in which $\tilde{z}$ is the only variable that appears: for t large enough, and for some $\alpha > 4$,

$$\|\tilde{z}(t)\|_{H^1} \leq C \left[\int_t^{+\infty} \frac{1}{s} \|\tilde{z}(s)\|_{H^1}\, ds + \frac{1}{t^{\alpha-2}} \int_t^{+\infty} \int_s^{+\infty} \|\tilde{z}(u)\|_{H^1}\, du\, ds \right] \tag{2.161}$$

(with C independent of z for the same reasons as those mentioned in Remark 2.9).

It results then $\tilde{z}(t) = 0$ in the neighborhood of $+\infty$. Note that one requires here $\|z(t)\|_{H^1} = \mathrm{O}\left(\frac{1}{t^\alpha}\right)$, with $\alpha > 4$, to hold. Consequently, uniqueness of a multi-soliton associated with the R_k, $k = 1, \ldots, K$ in the sense of (2.11) is proved also in the L^2-critical case.

Appendix

Linear independence of $\left(\partial_{x_i} Q_\omega\right)_{i \in \{1,\ldots,d\}}$

Proposition 2.30. *Let $u \in H^1(\mathbb{R}^d)$ be such that there exists $n \in \mathbb{R}^d \setminus \{0\}$ such that*

$$\forall\, x \in \mathbb{R}^d, \qquad n \cdot \nabla u(x) = 0.$$

Then $u = 0$.

Proof. Even if it means completing $\left\{\frac{n}{|n|}\right\}$ in an orthonormal basis of $\mathbb{R}^d$ and considering the passage matrix between the canonical basis and this new basis, we can always assume that n is the last vector of the canonical basis of $\mathbb{R}^d$. In that case, our assumption in Proposition 2.30 reads:

$$\forall\, (x_1, \ldots, x_d) \in \mathbb{R}^d, \qquad \partial_{x_d} u(x_1, \ldots, x_d) = 0,$$

or, in other words, for all $x_1, \ldots, x_{d-1} \in \mathbb{R}$, the application $x_d \mapsto u(x_1, \ldots, x_{d-1}, x_d)$ is constant, equal to $u(x_1, \ldots, x_{d-1}, 0)$.

Since $u \in L^2(\mathbb{R}^d)$, for all $x_1, \ldots, x_{d-1} \in \mathbb{R}$, one must have that

$$\int_{\mathbb{R}} |u(x_1, \ldots, x_{d-1}, x_d)|^2\, dx_d = \int_{\mathbb{R}} |u(x_1, \ldots, x_{d-1}, 0)|^2\, dx_d$$

is a finite quantity by Fubini theorem. This is the case if and only if $u(x_1, \ldots, x_{d-1}, 0) = 0$. Thus $u = 0$. □

Proof of Lemma 2.14

Assume that $1 \le k < j \le K$.
If $x_1 < \xi_{j-1} + (\sigma_{j-1} - A_0)t$, we have $\phi_j(t,x) = 0$.
If $x_1 \ge \xi_{j-1} + (\sigma_{j-1} - A_0)t$, then $x_1 > v_{k,1}t$ for large values of t and thus, by (2.5), we have

$$\begin{aligned}|R_k(t,x)| &\le Ce^{-\frac{\sqrt{\omega_k}}{4}(x_1-v_{k,1}t)}e^{-\frac{\sqrt{\omega_k}}{4}|x-v_kt|} \le Ce^{-\frac{\sqrt{\omega_k}}{4}(\sigma_{j-1}-A_0-v_{k,1})t}e^{-\frac{\sqrt{\omega_k}}{4}|x-v_kt|}\\ &\le Ce^{-\frac{\sqrt{\omega_k}}{4}(\sigma_{j-1}-A_0-v_{j-1,1})t}e^{-\frac{\sqrt{\omega_k}}{4}|x-v_kt|} \le Ce^{-\frac{\sqrt{\omega_k}}{4}\left(\frac{v_{j,1}-v_{j-1,1}}{2}-A_0\right)t}e^{-\frac{\sqrt{\omega_k}}{4}|x-v_kt|}\\ &\le Ce^{-\gamma t}e^{-\frac{\sqrt{\omega_k}}{4}|x-v_kt|}.\end{aligned}$$

Assume now that $1 \le j < k \le K$.
If $x_1 > \xi_j + (\sigma_j + A_0)t$, we have $\phi_j(t,x) = 0$.
If $x_1 \le \xi_j + (\sigma_j + A_0)t$, then $x_1 < v_{k,1}t$ for large values of t and thus we have as before

$$\begin{aligned}|R_k(t,x)| &\le Ce^{-\frac{\sqrt{\omega_k}}{4}(v_{k,1}t-x_1)}e^{-\frac{\sqrt{\omega_k}}{4}|x-v_kt|} \le Ce^{-\frac{\sqrt{\omega_k}}{4}(v_{k,1}-\sigma_j-A_0)t}e^{-\frac{\sqrt{\omega_k}}{4}|x-v_kt|}\\ &\le Ce^{-\frac{\sqrt{\omega_k}}{4}(v_{j+1,1}-\sigma_j-A_0)t}e^{-\frac{\sqrt{\omega_k}}{4}|x-v_kt|} \le Ce^{-\gamma t}e^{-\frac{\sqrt{\omega_k}}{4}|x-v_kt|}.\end{aligned}$$

Thus for all $k \ne j$,

$$|R_k(t,x)\phi_j(t,x)| \le Ce^{-\gamma t}e^{-\frac{\sqrt{\omega_k}}{4}|x-v_kt|}$$

and of course the same estimate is valid for $|\partial_{x_1}R_k(t,x)\phi_j(t,x)|$. This proves (2.89).

In a similar way, one proves (2.90). Now, let us show how to obtain (2.91). First, notice that it is sufficient to prove (2.91) with ψ_k instead of ϕ_k. Then,

$$\partial_{x_1}\psi_k(t,x) = \frac{1}{t}\psi'\left(\frac{x_1-\xi_k-\sigma_kt}{t}\right), \quad \partial^3_{x_1}\psi_k(t,x) = \frac{1}{t^3}\psi^{(3)}\left(\frac{x_1-\xi_k-\sigma_kt}{t}\right),$$

and

$$\partial_t\psi_k(t,x) = -\frac{x_1-\xi_k-\sigma_kt}{t^2}\psi'\left(\frac{x_1-\xi_k-\sigma_kt}{t}\right) - \frac{\sigma_k}{t}\psi'\left(\frac{x_1-\xi_k-\sigma_kt}{t}\right).$$

Hence,

$$|\partial_{x_1}\psi_k(t,x)| \le \frac{1}{t}\|\psi'\|_{L^\infty}, \quad |\partial^{(3)}_{x_1}\psi_k(t,x)| \le \frac{1}{t^3}\|\psi^{(3)}\|_{L^\infty},$$

and

$$|\partial_t\psi_k(t,x)| \le \frac{1}{t}\|x\psi'\|_{L^\infty} + \frac{\sigma_k}{t}\|\psi'\|_{L^\infty},$$

which leads to (2.91).

To finish with, let us observe that $\partial_{x_1}\phi_j(t,x) = 0$ if $\xi_{j-1} + (\sigma_{j-1} + A_0)t \le x_1 \le \xi_j + (\sigma_j - A_0)t$ (and in fact also if $x_1 \le \xi_{j-1} + (\sigma_{j-1} - A_0)t$ or $x_1 \ge \xi_j + (\sigma_j + A_0)t$). Thus, for $k \ne j$, the proof of (2.92) is just a copy of that of (2.89).
Moreover, if $x_1 \le \xi_{j-1} + (\sigma_{j-1} + A_0)t$, then $x_1 < v_{j,1}t$ for t large and thus, as before, we obtain

$$\begin{aligned}|R_j(t,x)| &\le Ce^{-\frac{\sqrt{\omega_j}}{4}(v_jt-x_1)}e^{-\frac{\sqrt{\omega_k}}{4}|x-v_jt|} \le Ce^{-\frac{\sqrt{\omega_j}}{4}(v_jt-\sigma_{j-1}-A_0)t}e^{-\frac{\sqrt{\omega_k}}{4}|x-v_jt|}\\ &\le Ce^{-\gamma t}e^{-\frac{\sqrt{\omega_k}}{4}|x-v_jt|}.\end{aligned}$$

If $x_1 \geq \xi_j + (\sigma_j - A_0)t$, then $x_1 > v_{j,1}t$ and one obtains again

$$|R_j(t,x)| \leq Ce^{-\gamma t}e^{-\frac{\sqrt{\omega_k}}{4}|x-v_jt|}.$$

Hence, using in addition (2.91), we deduce from what precedes that

$$|R_k(t,x)\partial_{x_1}\phi_j(t,x)| \leq C\frac{e^{-\gamma t}}{t}e^{-\frac{\sqrt{\omega_k}}{4}|x-v_jt|}.$$

In this manner, we obtain (2.92).

Proof of Proposition 2.25: coercivity property in the L^2-critical case

In the L^2-critical (pure power) case, we consider $f : r \mapsto r^{\frac{2}{d}}$ so that the linearized operators around Q_ω rewrite $L_{+,\omega}(v) = -\Delta v + \omega v - \left(1+\frac{4}{d}\right)Q_\omega^{\frac{4}{d}}v$ and $L_{-,\omega}(v) = -\Delta v + \omega v - Q_\omega^{\frac{4}{d}}v$.
Let us prove Proposition 2.25, following the results and ideas of Weinstein [108].

Due to Weinstein [108, Proposition 2.7], $\inf_{v\in H^1(\mathbb{R}^d),\langle Q_\omega,v\rangle=0}\langle L_{+,\omega}v,v\rangle = 0$ so that we can set $\tau := \inf_{v\in S}\langle L_{+,\omega},v\rangle$, where S is the set of all $v \in H^1(\mathbb{R}^d)$ such that $\langle Q_\omega, v\rangle = 0$, for all $i = 1,\dots,d$, $\langle\partial_{x_i}Q_\omega, v\rangle = 0$, $\langle x\cdot\nabla Q_\omega, v\rangle = 0$, and $\|v\|_{H^1} = 1$. We have obviously $\tau \geq 0$; we aim to show that $\tau > 0$.

Assume by contradiction that $\tau = 0$. Then for all $n \in \mathbb{N}$, there exists $v_n \in S$ such that $\langle L_{+,\omega}v_n, v_n\rangle \leq \frac{1}{n+1}$.
This implies

$$0 < \min\{\omega,1\}\|v_n\|_{H^1}^2 \leq \int_{\mathbb{R}^d}|\nabla v_n|^2\,dx + \omega\int_{\mathbb{R}^d}|v_n|^2\,dx \leq \left(1+\frac{4}{d}\right)\int_{\mathbb{R}^d}Q_\omega^{\frac{4}{d}}v_n^2\,dx + \frac{1}{n+1}. \tag{2.162}$$

In addition, $(\|v_n\|_{H^1})_n$ is uniformly bounded so that, up to extraction, (v_n) converges in $H^1(\mathbb{R}^d)$ for the weak topology, say to $v^* \in H^1(\mathbb{R}^d)$. And so, we have

$$\langle Q_\omega, v^*\rangle = 0, \quad \forall\, i\in\{1,\dots,d\},\ \langle\partial_{x_i}Q_\omega, v^*\rangle = 0, \quad \langle x\cdot\nabla Q_\omega, v^*\rangle = 0. \tag{2.163}$$

By means of Hölder inequality, interpolation, and exponential decay of Q_ω, (2.163) leads to

$$\int_{\mathbb{R}^d}Q_\omega^{\frac{4}{d}}v_n^2\,dx \underset{n\to+\infty}{\longrightarrow} \int_{\mathbb{R}^d}Q_\omega^{\frac{4}{d}}v^{*2}\,dx. \tag{2.164}$$

By passing to the limit as n tends to $+\infty$, it results from (2.162) and (2.164) that

$$0 < \min\{\omega,1\} \leq \left(1+\frac{4}{d}\right)\int_{\mathbb{R}^d}Q_\omega^{\frac{4}{d}}v^{*2}\,dx.$$

In particular, $v^* \neq 0$.
Moreover, by weak convergence, we have

$$\|v^*\|_{L^2} \leq \liminf_{n\to+\infty}\|v_n\|_{L^2} \qquad \text{and} \qquad \|\nabla v^*\|_{L^2} \leq \liminf_{n\to+\infty}\|\nabla v_n\|_{L^2}. \tag{2.165}$$

It follows from (2.164) and (2.165) that

$$0 \leq \langle L_{+,\omega} v^*, v^* \rangle \leq \liminf_{n \to +\infty} \langle L_{+,\omega} v_n, v_n \rangle = 0;$$

in other words, $\frac{v^*}{\|v^*\|_{H^1}}$ is an element of S which minimizes $v \mapsto \langle L_{+,\omega} v, v \rangle$. Even if it means considering $\frac{v^*}{\|v^*\|_{H^1}}$, we will assume moreover $\|v^*\|_{H^1} = 1$. We are thus led to the following Lagrange multiplier condition:

$$L_{+,\omega} v^* = \alpha v^* + \beta Q_\omega + \sum_{i=1}^{d} \gamma_i \partial_{x_i} Q_\omega + \delta x \cdot \nabla Q_\omega, \tag{2.166}$$

for some reals α, β, γ_i, and δ. Since $0 = \langle L_{+,\omega} v^*, v^* \rangle$, (2.163) implies that $\alpha = 0$. Then, for all $j \in \{1, \dots, d\}$,

$$0 = \langle L_{+,\omega} \partial_{x_j} Q_\omega, v^* \rangle = \langle L_{+,\omega} v^*, \partial_{x_j} Q_\omega \rangle = \sum_{i=1}^{d} \gamma_i \langle \partial_{x_i} Q_\omega, \partial_{x_j} Q_\omega \rangle.$$

Given that $\partial_{x_1} Q_\omega, \dots, \partial_{x_d} Q_\omega$ are linearly independent in $L^2(\mathbb{R}^d)$, the $d \times d$-matrix with entries $\langle \partial_{x_i} Q_\omega, \partial_{x_j} Q_\omega \rangle$ is invertible. Consequently, for all $i \in \{1, \dots, d\}$, $\gamma_i = 0$.
Now, using

$$0 = \langle -2Q_\omega, v^* \rangle, \qquad L_{+,\omega} \left(\frac{d}{2} Q_\omega + x \cdot \nabla Q_\omega \right) = -2Q_\omega$$

(which is specific to the critical case), and the symmetry of the bilinear form $\langle L_{+,\omega} \cdot, \cdot \rangle$, we deduce that

$$\begin{aligned} 0 = \left\langle L_{+,\omega} v^*, \frac{d}{2} Q_\omega + x \cdot \nabla Q_\omega \right\rangle &= \frac{\beta d}{2} \|Q_\omega\|_{L^2}^2 - \frac{\delta d^2}{4} - \frac{\beta d}{2} \|Q_\omega\|_{L^2}^2 + \delta \int_{\mathbb{R}^d} (x \cdot \nabla Q_\omega)^2 \, dx \\ &= \delta \left(\int_{\mathbb{R}^d} (x \cdot \nabla Q_\omega)^2 \, dx - \frac{d^2}{4} \int_{\mathbb{R}^d} Q_\omega^2 \, dx \right). \end{aligned}$$

But we have

$$\int_{\mathbb{R}^d} \left(x \cdot \nabla Q_\omega \right)^2 dx - \frac{d^2}{4} \int_{\mathbb{R}^d} Q_\omega^2 \, dx > 0, \tag{2.167}$$

considering that this quantity is nothing but the square of the L^2 norm of $\frac{d}{2} Q_\omega + x \cdot \nabla Q_\omega$ (and $\frac{d}{2} Q_\omega + x \cdot \nabla Q_\omega$ is obviously not zero).
Hence $\delta = 0$, and finally (2.166) reduces to $L_{+,\omega} v^* = \beta Q_\omega$.
We claim now that $\beta \neq 0$: otherwise (using the well-known non-degeneracy condition (2.17) of $L_{+,\omega}$ in the present case) v^* would be a linear combination of the $\partial_{x_i} Q_\omega$, $i = 1, \dots, d$, and then it would result $v^* = 0$ (since for all i, $\langle v^*, \partial_{x_i} Q_\omega \rangle = 0$), which is not the case.
Thus $v^* = -\frac{2}{\beta} \left(\frac{d}{2} Q_\omega + x \cdot \nabla Q_\omega \right)$ and

$$\begin{aligned} 0 = \langle v^*, x \cdot \nabla Q_\omega \rangle &= -\frac{2}{\beta} \left\langle \frac{d}{2} Q_\omega + x \cdot \nabla Q_\omega, x \cdot \nabla Q_\omega \right\rangle \\ &= -\frac{2}{\beta} \left(\int_{\mathbb{R}^d} (x \cdot \nabla Q_\omega)^2 \, dx - \frac{d^2}{4} \int_{\mathbb{R}^d} Q_\omega^2 \, dx \right), \end{aligned}$$

which contradicts (2.167). So we come to the conclusion that τ is positive; hence Proposition 2.25 is established.

Chapter 3

Non dispersive solutions of the generalized KdV equations are typically multi-solitons

Abstract
We consider solutions of the generalized Korteweg-de Vries equations (gKdV) which are non dispersive in some sense (in the spirit of [70]) and which remain close to multi-solitons. We show that these solutions are necessarily pure multi-solitons. For the Korteweg-de Vries equation (KdV) and the modified Korteweg-de Vries equation (mKdV) in particular, we obtain a characterization of multi-solitons and multi-breathers in terms of non dispersion.

3.1 Introduction

3.1.1 Setting of the problem and known results

We consider the generalized Korteweg-de Vries equations

$$\begin{cases} \partial_t u + \partial_x(\partial_x^2 u + u^p) = 0 \\ u(0) = u_0 \in H^1(\mathbb{R}) \end{cases} \tag{gKdV}$$

where (t, x) are elements of $\mathbb{R} \times \mathbb{R}$ and $p > 1$ is an integer.

Recall that the Cauchy problem for (gKdV) is locally well-posed in $H^1(\mathbb{R})$ from a standard result by Kenig, Ponce and Vega [50] and that the two following quantities are conserved for each solution u of (gKdV) for all t:

- (the L^2-mass) $\int_{\mathbb{R}} u^2(t,x)\ dx$
- (the energy) $\int_{\mathbb{R}} \left\{ \frac{1}{2} u_x^2 - \frac{1}{p+1} u^{p+1} \right\} (t,x)\ dx.$

Ce chapitre fait l'objet d'un article publié dans les *Annales de l'Institut Henri Poincaré C, Analyse non linéaire* [30].

In addition, the set of solutions of (gKdV) is conserved under scaling transformation

$$u \longmapsto \left((t,x) \mapsto \lambda^{\frac{2}{p-1}} u\left(\lambda^3 t, \lambda x\right)\right),$$

for all $\lambda > 0$, and the $\dot{H}^{\sigma(p)}$-norm is invariant under this transformation, where $\sigma(p) := \frac{1}{2} - \frac{2}{p-1}$. Let us recall that the global dynamics of the solutions depends on the sign of $\sigma(p)$. The case $\sigma(p) < 0$ that is $1 < p < 5$, is called L^2-subcritical, and all H^1-solutions of (gKdV) are then global (in time) and H^1-uniformly bounded. If $\sigma(p) = 0$, that is $p = 5$, we are in L^2-critical case and solutions might blow up in finite time [68,69,77–79]. In the L^2-supercritical case, corresponding to $\sigma(p) > 0$ (or $p > 5$), much less is known but finite time blow up is expected: existence of $p^* > 5$ and of blow-up solutions for all $p \in (5, p^*)$ are proven in [56].

Moreover it is well-known that (gKdV) admits a family of explicit traveling wave solutions indexed by $\mathbb{R}_+^* \times \mathbb{R}$. Let Q be the unique (up to translation) positive solution in $H^1(\mathbb{R})$ (known also as *ground state*) to the following stationary elliptic problem associated with (gKdV)

$$Q'' + Q^p = Q,$$

given by the explicit formula

$$Q(x) = \left(\frac{p+1}{2\mathrm{ch}^2\left(\frac{p-1}{2}x\right)}\right)^{\frac{1}{p-1}}.$$

Then for all $c_0 > 0$ (velocity parameter) and $x_0 \in \mathbb{R}$ (translation parameter),

$$R_{c_0,x_0}(t,x) = Q_{c_0}(x - c_0 t - x_0) \tag{3.1}$$

is a global traveling wave solution of (gKdV) classically named *soliton* solution, where $Q_{c_0}(x) = c_0^{\frac{1}{p-1}} Q(\sqrt{c_0} x)$. It is orbitally stable if and only if $p < 5$ (L^2-subcritical case) (see Weinstein [109], Bona, Souganidis and Strauss [3], Grillakis, Shatah and Strauss [37], and Martel and Merle [67]).

Solitons are special objects which enjoy very specific properties. Let us recall the following rigidity result, which roughly asserts that non dispersive solutions to (gKdV) which are close to solitons are actually exactly solitons.

Theorem 3.1 (Liouville property near a soliton; Martel and Merle [65, 72])**.** *Let $c_0 > 0$. There exists $\alpha > 0$ such that if $u \in \mathscr{C}(\mathbb{R}, H^1(\mathbb{R}))$ is a solution of (gKdV) satisfying, for some $\mathscr{C}^1$ function $y : \mathbb{R} \to \mathbb{R}$,*

$$\textit{(closeness to a soliton)} \qquad \forall\, t \in \mathbb{R}, \qquad \|u(t, \cdot + y(t)) - Q_{c_0}\|_{H^1} \le \alpha, \tag{3.2}$$

$$\textit{(non dispersion)} \qquad \forall\, \varepsilon > 0,\ \exists\, R > 0,\ \forall\, t \in \mathbb{R}, \qquad \int_{|x|>R} u^2(t, x + y(t))\, dx \le \varepsilon, \tag{3.3}$$

then there exist $c_1 > 0$, $x_1 \in \mathbb{R}$ such that

$$\forall\, t, x \in \mathbb{R}, \qquad u(t,x) = Q_{c_1}(x - x_1 - c_1 t).$$

This striking result has its own interest of course, but we emphasize that it is also a key ingredient to prove asymptotic stability of (gKdV) solitons (we refer to [65, 66, 72]). We highlight the fact that this result applies in each mass subcritical, critical, and supercritical case by requiring the solution u to remain close to a soliton (up to translation) for all times (3.2). In the L^2-subcritical case where solitons are known to be stable, (3.2) can be relaxed to hold only at $t = 0$.

Finally let us note that solitons play a fundamental role in the study and the understanding of the (gKdV) flow; the important soliton resolution conjecture asserts that any solution with generic initial condition behaves as a sum of solitons plus a radiative-dispersive term as time goes to infinity. In this spirit, built upon solitons, we are interested in other solutions to our problem, namely multi-soliton solutions, defined as follows.

Definition 3.2. *Let $N \geq 1$ and consider N solitons R_{c_i,x_i} as in (3.1) with speeds $0 < c_1 < \cdots < c_N$. A multi-soliton in $+\infty$ (resp. in $-\infty$) associated with the R_{c_i,x_i} is an H^1-solution u of (gKdV) defined in a neighborhood of $+\infty$ (resp. $-\infty$) and such that*

$$\left\| u(t) - \sum_{i=1}^{N} R_{c_i,x_i}(t) \right\|_{H^1} \to 0, \qquad \text{as } t \to +\infty \text{ (resp. as } t \to -\infty). \tag{3.4}$$

Multi-solitons are known to exist for all $p > 1$; they are even explicit for $p = 2$ (KdV) [89, section 16] and for $p = 3$ (mKdV) [101, Chapter 5, formula (5.5)]. What is more, the classification of the multi-solitons of (gKdV) is complete. Let us gather the main results.

Theorem 3.3 (Martel [63]; Côte, Martel and Merle [19]; Combet [11])**.** *Let $p > 1$ be an integer and let $N \geq 1$, $0 < c_1^+ < \cdots < c_N^+$, and $x_1^+, \ldots, x_N^+ \in \mathbb{R}$.*
If $p \leq 5$, there exists $T_0 \geq 0$ and a unique multi-soliton $u \in \mathscr{C}([T_0, +\infty), H^1(\mathbb{R}))$ associated with the $R_{c_i^+,x_i^+}$, $i \in \{1, \ldots, N\}$.
If $p > 5$, there exists a one-to-one map Φ from $\mathbb{R}^N$ to the set of all H^1-solutions of (gKdV) defined in a neighborhood of $+\infty$ such that u is a multi-soliton in $+\infty$ associated with the $R_{c_i^+,x_i^+}$ if and only if there exist $\lambda \in \mathbb{R}^N$ and $T_0 \geq 0$ such that $u_{|[T_0,+\infty)} = \Phi(\lambda)_{|[T_0,+\infty)}$.
Moreover, in each case, u belongs to $\mathscr{C}([T_0, +\infty), H^s(\mathbb{R}))$ for all $s \geq 0$, and there exist $\theta > 0$ and positive constants C_s such that for all $s \geq 0$, for all $t \geq T_0$,

$$\left\| u(t) - \sum_{i=1}^{N} R_{c_i^+,x_i^+}(t) \right\|_{H^s} \leq C_s e^{-\theta t}. \tag{3.5}$$

In the L^2-subcritical case (like solitons), sums of decoupled and ordered solitons are stable in $H^1(\mathbb{R})$, even asymptotically stable (Martel, Merle and Tsai [80] and Martel and Merle [72]), and so are multi-solitons.

3.1.2 Main results

Several properties available for solitons have been adapted or even extended to multi-solitons. This article precisely takes this step since it aims at providing an analogue of the rigidity property of Theorem 3.1 in the multi-soliton case. We consider solutions of (gKdV) that are non dispersive in some sense and uniformly close to the sum of N solitons, and show that they are exact multi-solitons.

Theorem 3.4 (Liouville property near a multi-soliton)**.** *Let u be a solution of (gKdV) which belongs to $\mathscr{C}([0,+\infty), H^1(\mathbb{R}))$. Assume the existence of $\rho > 0$ such that*

$$\forall\, \varepsilon > 0,\ \exists\, R_\varepsilon > 0,\ \forall\, t \geq 0, \qquad \int_{x<\rho t - R_\varepsilon} u^2(t,x)\ dx \leq \varepsilon. \tag{3.6}$$

Let $N \geq 1$ be an integer and consider N positive real numbers $0 < c_1 < \cdots < c_N$. There exists $\alpha = \alpha(c_1, \dots, c_N, \rho) > 0$ such that the following holds: if there exist N functions $x_1, \dots, x_N$: $\mathbb{R}^+ \to \mathbb{R}$ of class $\mathscr{C}^1$ satisfying

$$\forall\, t \geq 0, \qquad \left\| u(t) - \sum_{i=1}^{N} Q_{c_i}(\cdot - x_i(t)) \right\|_{H^1} \leq \alpha, \tag{3.7}$$

and

$$\forall\, t \geq 0,\ \forall\, i \in \{1, \dots, N-1\}, \qquad x_{i+1}(t) - x_i(t) \geq |\ln \alpha|, \tag{3.8}$$

then u is a multi-soliton (in $+\infty$). In other words, there exist $\theta > 0$, $0 < c_1^+ < \cdots < c_N^+$, $x_1^+, \dots, x_N^+ \in \mathbb{R}$ and positive constants C_s such that for all $t \geq 0$, (3.5) is granted.

Remark 3.5. Assumption (3.8) is done so that solitons are sufficiently decoupled, and thus do not collide. In the L^2-subcritical case $1 < p < 5$, as sum of decoupled solitons are stable, assumptions (3.7) and (3.8) can be relaxed to hold only at time $t = 0$.

This result is a natural extension of Theorem 3.1 to multi-solitons in $+\infty$, which are the only solutions which are non dispersive in the sense (3.6) (and remain close to a sum of solitons): this is a nice dynamical characterization of multi-solitons among solutions to (gKdV). By contraposition, it means that if a solution u remains in large time sufficiently close to a multi-soliton but is *not* a multi-soliton, then it disperses insofar as (3.6) fails.

We emphasize that, in contrast with the original statement for one soliton (Theorem 3.1), the non dispersion assumption (3.6) requires the mass to be located essentially for $x \geq \rho t$ for some small positive speed $\rho > 0$ (and almost touches $x = 0$); it allows (seemingly) for much more room than in the condition (3.3), which requires that the mass be essentially concentrated in a moving ball of fixed size R_ε. Note that this non dispersion property (3.6) is equivalent to the following assertion:

$$\exists\, \rho > 0, \qquad \int_{x \leq \rho t} u^2(t,x)\ dx \to 0 \quad \text{as } t \to +\infty. \tag{3.9}$$

(The equivalence between the two assertions is shown to hold in the Appendix.)

Furthermore, the assumptions in Theorem 3.4 are done only for positive times $t \geq 0$ (and not for all times $t \in \mathbb{R}$). As it applies of course to the case of a single soliton, Theorem 3.4 actually extends and refines Theorem 3.1.

We must underline that this improvement to focus on the behavior for positive times only is actually very meaningful. Indeed, in view of the above result, a solution which would be non dispersive at times $+\infty$ and $-\infty$ would be a multi-soliton at both ends: but such a behavior is not to be expected, except in the integrable cases of $p = 2$ (KdV) and $p = 3$ (mKdV) and the Gardner nonlinearity $u^2 - \lambda u^3$. To support this, let us refer to the work by Martel and Merle [73–75] (see also Muñoz [90]) on the description of 2-solitons: starting with a 2-soliton solution at $-\infty$ (for the quartic $p = 4$ (gKdV)), the collision is almost but not elastic, and there is a non zero defect (which one can quantify), so that it is not a 2-soliton at $+\infty$ (and so, by Theorem 3.4, it must be dispersive in the sense of (3.6)).

In principle, the computations in the articles above could extend to N-solitons for $N \geq 3$, but it has not been performed yet, and one could still wonder if there is always a defect. If one is willing to assume non dispersion for all time $t \in \mathbb{R}$, our conclusion is that the solution under consideration is a multi-soliton in $+\infty$ for which all derivatives decay exponentially in space for each fixed values of t. More precisely, we have

Corollary 3.1. *Let $u \in \mathscr{C}(\mathbb{R}, H^1(\mathbb{R}))$ satisfy the assumptions (3.7), (3.8) of Theorem 3.4, and assume (to replace (3.6)) the existence of two constants $0 < \sigma < \rho$ such that*

$$\forall\, \varepsilon > 0,\ \exists\, R_\varepsilon > 0,\ \forall\, t \in \mathbb{R}, \qquad \int_{\mathscr{B}(\rho t, \sigma|t|+R_\varepsilon)^{\complement}} u^2(t,x)\, dx \leq \varepsilon. \tag{3.10}$$

Then the conclusion of Theorem 3.4 holds, and also the following exponential decay property at fixed time, for all $s \in \mathbb{N}$, and for some possibly larger constant C_s:

$$\forall\, t \geq 0,\ \forall\, x \in \mathbb{R}, \qquad \left|\partial_x^s u(t,x)\right| \leq C_s \sum_{i=1}^{N} e^{-\theta|x-c_i^+ t|}. \tag{3.11}$$

As it was first observed in [65], non dispersion (for all times) actually self improves to smoothness and exponential decay in space (outside the center of mass). Of course, this is very relevant for solitons, which exhibit precisely this spatial behavior. But it has yet to be proven that multi-solitons do have spatial exponential decay (3.11) as well; even though it is a very natural conjecture, and that it is known that multi-solitons are smooth. To be able to conclude to (3.11), one has currently to make the assumption (3.10) (in fact, it would be sufficient to assume that u and $u(-t)$ satisfy (3.6) and to assume in addition that the analog of (3.6) with $x > (\rho + \sigma)t + R_\varepsilon$ holds for positive times), and for the time being, the above Corollary 3.1 is meaningful.

Remark 3.6. In the L^2-subcritical case $1 < p < 5$, assumptions (3.7) and (3.8) can be relaxed to hold only a time $t = 0$. If they hold for large enough times, positive and negative (or outside of the collision period), the conclusion can be strengthened to u being a multi-soliton at $+\infty$ and $-\infty$, and satisfying (3.11) for all $t \in \mathbb{R}$.

In the context of the particular (KdV) equation (corresponding to $p = 2$), we claim next a result which gives rise to a simplified characterization of multi-solitons among all H^1-solutions.

Theorem 3.7. *Let $p = 2$ and $u_0 \in \mathscr{S}(\mathbb{R}) \setminus \{0\}$ be such that the corresponding solution u of (KdV), which is defined globally in time, is non dispersive for positive times, that is, satisfies (3.6). Then u is a multi-soliton (in $+\infty$ and $-\infty$).*

The proof of this theorem relies on the soliton resolution result for (KdV), set up by Eckhaus and Schuur [28] and refined in Schuur [101].

Remark 3.8. Requiring that the initial condition u_0 belongs to the Schwartz space is not necessary in order to reach the conclusion in Theorem 3.7. Considering the non dispersion assumption made in Theorem 3.7, it would be sufficient for example that all derivatives up to order 4 of u_0 decay faster than x^{-11} when $x \to +\infty$. Actually, we only need to assume that u_0 is smooth enough and decays sufficiently rapidly for $|x| \to +\infty$ for the whole of the inverse scattering method to work, thus for the soliton resolution result for (KdV) to hold [9, 101]. However, our goal is not to obtain the most general statement, and for clarity purposes, we will not attempt to optimize the regularity and decay assumptions on u_0.

Similarly, we can characterize non dispersive solutions of the (mKdV) equation. Recall that, in addition to solitons, (mKdV) admits other particular solutions, known as breathers, which are also important with respect to the soliton resolution conjecture. Breathers do not correspond to a superposition of solitons but are instead periodic in time and can move both in the left and right directions; for all $(\alpha, \beta) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$, and for all $x_1, x_2 \in \mathbb{R}$, the breather B_{α,β,x_1,x_2} with envelope velocity $\gamma := \beta^2 - 3\alpha^2$, phase velocity $\delta := 3\beta^2 - \alpha^2$, and translation parameters $x_1, x_2 \in \mathbb{R}$ takes the following expression:

$$B_{\alpha,\beta,x_1,x_2}(t,x) := 2\sqrt{2}\partial_x \left[\arctan\left(\frac{\beta}{\alpha}\frac{\sin(\alpha(x-\delta t - x_1))}{\cosh(\beta(x-\gamma t - x_2))}\right)\right]. \tag{3.12}$$

We refer to Alejo and Muñoz [1] for the introduction and the study of stability in H^2 of these solutions. Note that the decomposition result in terms of solitons and breathers available for (mKdV) solutions and stated in [101, Chapter 5, Theorem 5.1] and more recently in [8, Theorem 1.10] holds under the assumption that the initial data u_0 is *generic* in the following sense: the set of all $\xi \in \mathbb{C}$ such that the classical Jost solutions $\psi_l(\xi)$ and $\psi_r(\xi)$ to the Zakharov-Shabat system

$$\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}' = \begin{pmatrix} -i\xi & u_0 \\ -u_0 & i\xi \end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$$

are $\mathbb{R}$-linearly dependent is finite and consists in the *scattering data*

$$\{i\sqrt{c_1}, \dots, i\sqrt{c_{N_1}}, \alpha_1 + i\beta_1, \dots, \alpha_{N_2} + i\beta_{N_2}\}, \tag{3.13}$$

with $N_1, N_2 \in \mathbb{N}$, $c_1, \dots, c_{N_1}, \alpha_1, \dots, \alpha_{N_2}, \beta_1, \dots, \beta_{N_2} \in \mathbb{R}_+^*$ such that

$$c_1 < \cdots < c_N, \qquad \beta_1^2 - 3\alpha_1^2 < \cdots < \beta_{N_2}^2 - 3\alpha_{N_2}^2, \tag{3.14}$$

and

$$\forall\ (i,j) \in \{1,\dots,N_1\} \times \{1,\dots,N_2\}, \quad c_i \neq \beta_j^2 - 3\alpha_j^2. \tag{3.15}$$

We refer to Schuur [101, Chapter 4], Chen and Liu [8, Paragraph 1.2], and the references therein for more details concerning genericity.

Our result on non dispersive solutions of (mKdV) writes as follows.

Theorem 3.9. *Let $p = 3$ and $u_0 \in \mathscr{S}(\mathbb{R}) \setminus \{0\}$ be generic (in the above sense) with scattering data* (3.13)*, and such that the corresponding global solution u of (mKdV) is non dispersive for positive times (that is, satisfies* (3.6)*).*

Then u is a multi-breather with positive speeds in $+\infty$: we have $N_1 + N_2 \geq 1$ and for all $j = 1, \dots, N_2$,

$$\beta_j^2 - 3\alpha_j^2 > 0,$$

and there exist $\gamma > 0$, positive constants C_s, signs $\epsilon_i = \pm 1$, and real parameters $x_{0,i}, x_{1,j}, x_{2,j}$ such that for all $s \geq 0$, u belongs to $\mathscr{C}([0,+\infty), H^s(\mathbb{R}))$ and

$$\forall\ t \geq 0, \qquad \left\| u(t) - \sum_{i=1}^{N_1} \epsilon_i R_{2c_i,x_{0,i}}(t) - \sum_{j=1}^{N_2} B_{\sqrt{2}\alpha_j,\sqrt{2}\beta_j,x_{1,j},x_{2,j}}(t) \right\|_{H^s} \leq C_s e^{-\gamma t}.$$

The proof is done by adapting that of Theorem 3.7 by writing the soliton/breather resolution for $p = 3$, and then using smoothness and uniqueness of multi-breathers and the estimates in higher order Sobolev spaces proved by Semenov [102].

Remark 3.10. Let us notice that Remark 3.8 applies also in the context of Theorem 3.9.

3.1.3 Comments

The proofs of Theorem 3.4 and Corollary 3.1 are in the spirit of the original Liouville result by Martel and Merle [65], and also the work of Laurent and Martel [57] on smoothness and decay of non dispersive solutions. An important ingredient is the observation that a crucial monotonicity formula holds under a much relaxed non dispersion assumption than previously made, see Proposition 3.11, which has its own interest. Also we underline a subtle but key difference in the strategy of the proof: we crucially rely at some point on the asymptotic stability of multi-solitons in the energy space (from [70], stated in Theorem 3.15). But let us recall this result itself is a consequence of the rigidity result for one soliton stated in Theorem 3.1: in some sense, the roles are reversed here.

In fact, our proofs use and combine several previous results on the (gKdV) flow around solitons and multi-solitons. This sheds a new light on many results established so far and which have their own interest, which are here linked together to yield new statements. It seems to us that this phenomenon is an interesting point of this paper.

Our results lead to several open questions. We already mentioned above the first one, but repeat it here: we conjecture that, for each time where defined, multi-solitons for (gKdV) have pointwise exponential decay (along with their derivatives); this is only known in the integrable case, where explicit formulas are known. As second question is whether similar rigidity results (as well as asymptotic stability properties) hold for other dispersive models. The Liouville theorem for solitons holds for the Zakharov-Kuznetsov equation in 2D for example, it would be nice to know if an analog for multi-solitons holds as well. A very natural context is that of the non-linear Schrödinger equations, for which the understanding of non dispersive solutions remains mostly open.

This article is organized as follows. After the introduction, we present in section 3.2 a general property of exponential decay satisfied by non dispersive solutions which is an important new observation and interesting in itself. The third section is then devoted to the proof of Theorem 3.4 and Corollary 3.1. In the forth section, we consider the integrable case and sketch the proofs of Theorems 3.7 and 3.9. In the Appendix, we are concerned with a proposition which is an ingredient in the proof of Theorem 3.4 and we justify the equivalence of the non dispersion property (3.6) with (3.9).

3.1.4 Acknowledgments

The author would like to thank his supervisor Raphaël Côte for suggesting the idea of this work and for fruitful discussions.

3.2 Smoothness and exponential decay for non dispersive solutions

The goal of this section is to show the following propositions which extend Laurent and Martel [57, Theorem 1].

Proposition 3.11. *Let J be a neighborhood of $+\infty$ and $u \in \mathscr{C}(J, H^1(\mathbb{R}))$ be a solution of (gKdV)* *which belongs also to $L^\infty(J, H^1(\mathbb{R}))$. Suppose that there exists $\rho > 0$ such that*

$$\forall\, \varepsilon > 0,\ \exists\, R_\varepsilon > 0,\ \forall\, t \in J, \qquad \int_{x<\rho t - R_\varepsilon} u^2(t,x)\, dx \le \varepsilon. \tag{3.16}$$

Then u belongs to $\mathscr{C}^\infty(J \times \mathbb{R})$ and there exists $\kappa > 0$ such that for all $k \in \mathbb{N}$, there exists $K_k > 0$ such that

$$\forall\, t \in J,\ \forall\, x < \rho t, \qquad \left|\partial_x^k u(t,x)\right| \le K_k e^{-\frac{\kappa}{2}|x-\rho t|}. \tag{3.17}$$

We state next another generalized version, which is useful in the proof of Corollary 3.1.

Proposition 3.12. *Let J be a neighborhood of $+\infty$ and $u \in \mathscr{C}(J, H^1(\mathbb{R}))$ be a solution of (gKdV) which belongs also to $L^\infty(J, H^1(\mathbb{R}))$. Suppose that there exist $\beta, \delta > 0$ and two $\mathscr{C}^1$ functions $a, b : J \to \mathbb{R}$ such that*

$$\forall\, t \in J, \qquad \delta \le a'(t) \le b'(t) \le \beta, \tag{3.18}$$

and

$$\forall\, \varepsilon > 0,\ \exists\, R_\varepsilon > 0,\ \forall\, t \in J, \qquad \int_{x<m(t) - R_\varepsilon} u^2(t,x)\, dx \le \varepsilon, \tag{3.19}$$

where $m(t) := \min\{a(t), b(t)\}$.
Then u belongs to $\mathscr{C}^\infty(J \times \mathbb{R})$ and there exists $\kappa > 0$ such that for all $k \in \mathbb{N}$, there exists $K_k > 0$ such that

$$\forall\, t \in J,\ \forall\, x < m(t), \qquad \left|\partial_x^k u(t,x)\right| \le K_k e^{-\frac{\kappa}{2}|x-m(t)|}. \tag{3.20}$$

Remark 3.13. It is to be noticed that, if J in Proposition 3.12 is replaced by a neighborhood J' of $-\infty$, then we conclude with an estimate at the right of $M(t) := \max\{a(t), b(t)\}$, or more precisely with the existence of $\kappa > 0$ such that for all $k \in \mathbb{N}$, there exists $K_k > 0$ such that

$$\forall\, t \in J',\ \forall\, x > M(t), \qquad \left|\partial_x^k u(t,x)\right| \le K_k e^{-\frac{\kappa}{2}(x-M(t))}. \tag{3.21}$$

This is justified by the following symmetry property for (gKdV) and the assumption in Proposition 3.12. Denoting $\hat{u}(t,x) := u(-t,-x)$, $\hat{a}(t) := -a(-t)$, $\hat{b}(t) := -b(-t)$, $\hat{m}(t) := -m(-t)$, and $\hat{M}(t) := -M(-t)$, we observe that u satisfies the assumptions of Proposition 3.11 on a neighborhood J' of $-\infty$ if and only if $\hat{u}$ satifies the same assumptions on $-J'$ (which is a neighborhood of $+\infty$) with a, b, m, and M replaced respectively by $\hat{a}$, $\hat{b}$, $\hat{M}$, and $\hat{m}$ in Proposition 3.12. Thus once we have proved (3.20) as stated in Proposition 3.12, we have immediately the pointwise estimate on $\hat{u}(t)$ at the left of $\hat{m}(t)$ for $t \in -J'$, which precisely provides (3.21), that is the desired pointwise estimate on $u(t)$ at the right of $M(t)$ for $t \in J'$.
Obviously, Propositions 3.11 and 3.12 apply in particular with $J = \mathbb{R}$, in which case both estimates (3.20) and (3.21) hold.

Now, proceeding essentially as Laurent and Martel [57, Theorem 1], we derive the proof of Proposition 3.12.

Proof. Step 1: Estimates to be established

By the classical Sobolev embedding $H^1(-\infty, m(t)) \hookrightarrow L^\infty(-\infty, m(t))$, it suffices to see that

$$\exists\, K > 0,\ \forall\, t \in J, \qquad \int_{x<m(t)} \left(u^2(t,x) + u_x^2(t,x)\right) e^{\kappa(m(t)-x)}\, dx \le K \tag{3.22}$$

holds to have the desired conclusion, that is (3.20), for $k = 0$ and for almost every $x < m(t)$. Using that $u(t)$ is continuous on $\mathbb{R}$ by $H^1(\mathbb{R}) \hookrightarrow \mathscr{C}(\mathbb{R})$, we deduce that (3.20) is true for $k = 0$. Similarly, to reach the whole conclusion, we have to show that for each $k \in \mathbb{N}$, there exists $\tilde{K}_k > 0$ such that

$$\forall\, t \in J, \qquad \int_{x<m(t)} \left(\partial_x^k u(t,x)\right)^2 e^{\kappa(m(t)-x)}\, dx \leq \tilde{K}_k. \tag{3.23}$$

In order to prove (3.23), it is convenient to introduce a well-chosen $\mathscr{C}^1$ function defined on J, denoted by $\tilde{m}$, which replaces somehow m in the case where m is not already $\mathscr{C}^1$. By this means, we get around the difficulty of a possible point where m is not differentiable. This is the purpose of the following:

Claim 3.14. *There exists $\tilde{m} : J \to \mathbb{R}$ of class $\mathscr{C}^1$ such that for all $t \in J$, $m(t) \leq \tilde{m}(t) \leq m(t) + 1$, and $\tilde{m}'(t) \geq \delta$.*

Proof of Claim 3.14. Define $\tilde{m}$ by

$$\forall\, t \in J, \quad \tilde{m}(t) := 1 + \frac{a(t)+b(t)}{2} - \sqrt{1 + \left(\frac{b(t)-a(t)}{2}\right)^2}.$$

Then $\tilde{m}$ is $\mathscr{C}^1$ on J and given that $\min\{a,b\} = \frac{a+b}{2} - \frac{|a-b|}{2}$, one can check that $\tilde{m}$ satisfies $m(t) \leq \tilde{m}(t) \leq m(t) + 1$ by means of the well-known inequality

$$\sqrt{x+y} \leq \sqrt{x} + \sqrt{y}, \qquad \text{valid for all } x, y \geq 0.$$

Moreover, by a straightforward computation, we have

$$\tilde{m}'(t) \geq \frac{a'(t)+b'(t)}{2} - \frac{b'(t)-a'(t)}{2} \geq a'(t).$$

Consequently, Claim 3.14 is proved. □

Now, we consider $\tilde{m}$ as in the previous claim. Judging by the fact that for all $t \in J$, $m(t) \leq \tilde{m}(t)$, we can write for all $k \in \mathbb{N}$

$$\int_{x<m(t)} \left(\partial_x^k u(t,x)\right)^2 e^{\kappa(m(t)-x)}\, dx \leq \int_{x<\tilde{m}(t)} \left(\partial_x^k u(t,x)\right)^2 e^{\kappa(\tilde{m}(t)-x)}\, dx. \tag{3.24}$$

Thus, to achieve our goal (3.23), it suffices to show the existence of $C_k > 0$ such that

$$\forall\, t \in J, \quad \int_{x<\tilde{m}(t)} \left(\partial_x^k u(t,x)\right)^2 e^{\kappa(\tilde{m}(t)-x)}\, dx \leq C_k. \tag{3.25}$$

Step 2: Proof of (3.25) *for $k = 0$*

We will obtain (3.25) by a strong monotonicity property which is the purpose of Lemma 3.2 and Lemma 3.3 below.

Let us introduce, for some $\kappa > 0$ to be determined later, the function φ defined by

$$\varphi(x) = \frac{1}{2} - \frac{1}{\pi}\arctan(e^{\kappa x}).$$

It satisfies the following properties

$$\exists\, \lambda_0 > 0,\ \forall\, x \in \mathbb{R}, \qquad \lambda_0 e^{-\kappa|x|} < -\varphi'(x) < \frac{1}{\lambda_0} e^{-\kappa|x|}, \tag{3.26}$$

$$\forall\, x \in \mathbb{R}, \qquad |\varphi^{(3)}(x)| \leq -\kappa^2 \varphi'(x). \tag{3.27}$$

$$\exists\, \lambda_1 > 0,\ \forall\, x \geq 0, \qquad \lambda_1 e^{-\kappa x} \leq \varphi(x). \tag{3.28}$$

Moreover, let us observe that

$$\int_{x<\tilde{m}(t)} u^2(t,x) e^{\kappa(\tilde{m}(t)-x)}\, dx = \int_{x<0} u^2\left(t, x+\tilde{m}(t)\right) e^{-\kappa x}\, dx, \tag{3.29}$$

and that, for all $x_0 < 0$,

$$\begin{aligned}\int_{x_0 \leq x < 0} u^2\left(t, x+\tilde{m}(t)\right) e^{-\kappa x}\, dx &\leq e^{-\kappa x_0} \int_{x \geq x_0} u^2\left(t, x+\tilde{m}(t)\right) e^{-\kappa(x-x_0)}\, dx \\ &\leq \frac{1}{\lambda_1} e^{-\kappa x_0} \int_{x \geq x_0} u^2\left(t, x+\tilde{m}(t)\right) \varphi(x-x_0)\, dx \\ &\leq \frac{1}{\lambda_1} e^{-\kappa x_0} \int_{\mathbb{R}} u^2\left(t, x+\tilde{m}(t)\right) \varphi(x-x_0)\, dx.\end{aligned} \tag{3.30}$$

By Claim 3.14, for all $t \in J$, $\tilde{m}'(t) \geq \delta$. Therefore there exists $\eta > 0$ and an increasing affine function $f : \mathbb{R} \to \mathbb{R}$ such that

$$\forall\, t \in J, \quad -f'(t) + \tilde{m}'(t) \geq \eta. \tag{3.31}$$

Now, for fixed $t_0 \in J$ and x_0 in $\mathbb{R}$, consider

$$\begin{array}{rccl} I_{(t_0,x_0)} : & J & \longrightarrow & \mathbb{R}^+ \\ & t & \longmapsto & \displaystyle\int_{\mathbb{R}} u^2(t, x+\tilde{m}(t)) \varphi\big(x - x_0 + f(t) - f(t_0)\big)\, dx. \end{array}$$

We have

$$\forall\, t \in J, \qquad I_{(t_0,x_0)}(t) = \int_{\mathbb{R}} u^2(t,x) \varphi\big(x - x_0 + f(t) - f(t_0) - \tilde{m}(t)\big)\, dx, \tag{3.32}$$

so that by differentiation with respect to t, we obtain

$$\begin{aligned}\frac{dI_{(t_0,x_0)}}{dt}(t) =& -3 \int_{\mathbb{R}} u_x^2(t,x) \varphi'(\tilde{x})\, dx - \big(-f'(t) + \tilde{m}'(t)\big) \int_{\mathbb{R}} u^2(t,x) \varphi'(\tilde{x})\, dx \\ &+ \int_{\mathbb{R}} u^2(t,x) \varphi^{(3)}(\tilde{x})\, dx + \frac{2p}{p+1} \int_{\mathbb{R}} u^{p+1}(t,x) \varphi'(\tilde{x})\, dx,\end{aligned} \tag{3.33}$$

where $\tilde{x} := x - x_0 + f(t) - f(t_0) - \tilde{m}(t)$.
Set $\kappa := \sqrt{\frac{\eta}{2}}$. We claim then

Lemma 3.2. *There exists $C_0 > 0$ such that for all $x_0 \in \mathbb{R}$, and for all $t_0, t \in J$,*

$$\frac{dI_{(t_0,x_0)}}{dt}(t) \geq -C_0 e^{-\kappa\left(-x_0 + f(t) - f(t_0)\right)}. \tag{3.34}$$

Proof of Lemma 3.2. Due to the choice of κ and property (3.27) of φ, we have

$$\left|\int_{\mathbb{R}} u^2(t,x)\varphi^{(3)}(\tilde{x})\ dx\right| \leq -\frac{\eta}{2}\int_{\mathbb{R}} u^2(t,x)\varphi'(\tilde{x})\ dx. \tag{3.35}$$

Furthermore we control the non-linear part by considering, for $R > 0$,

$$I_1(t) := \int_{|\tilde{x}|>-x_0-R+f(t)-f(t_0)} u^{p+1}(t,x)\varphi'(\tilde{x})\ dx$$

and

$$I_2(t) := \int_{\mathbb{R}} u^{p+1}(t,x)\varphi'(\tilde{x})\ dx - I_1(t).$$

On the one hand, we have due to (3.26)

$$\left|I_1(t)\right| \leq \frac{1}{\lambda_0}e^{-\kappa\left(-x_0-R+f(t)-f(t_0)\right)}\left(\int_{\mathbb{R}}|u|^{p+1}(t,x)\ dx\right) \leq Ce^{-\kappa\left(-x_0-R+f(t)-f(t_0)\right)},$$

where we have used the Sobolev embedding $H^1(\mathbb{R}) \hookrightarrow L^{p+1}(\mathbb{R})$ and the fact that u belongs to $L^\infty(J, H^1(\mathbb{R}))$. Note that $C > 0$ is independent of x_0, t_0, and t.
On the other, we observe that if $|\tilde{x}| \leq -x_0 - R + f(t) - f(t_0)$, then $x \leq \tilde{m}(t) - R$ in particular, and therefore by Claim 3.14 we have also $x \leq m(t) - R + 1$. Thus, it follows

$$\begin{aligned}\left|I_2(t)\right| &\leq \|u(t)\|^{p-1}_{L^\infty\left(x\leq m(t)-R+1\right)}\int_{x\leq m(t)-R+1} u^2(t,x)|\varphi'(\tilde{x})|\ dx\\ &\leq \sqrt{2}^{p-1}\|u(t)\|^{\frac{p-1}{2}}_{L^2\left(x\leq m(t)-R+1\right)}\|u_x(t)\|^{\frac{p-1}{2}}_{L^2\left(x\leq m(t)-R+1\right)}\int_{\mathbb{R}} u^2(t,x)|\varphi'(\tilde{x})|\ dx\\ &\leq \sqrt{2}^{p-1}\|u(t)\|^{\frac{p-1}{2}}_{L^2\left(x\leq m(t)-R+1\right)}\sup_{t\in\mathbb{R}}\|u(t)\|^{\frac{p-1}{2}}_{H^1}\int_{\mathbb{R}} u^2(t,x)|\varphi'(\tilde{x})|\ dx.\end{aligned} \tag{3.36}$$

By the non dispersion assumption (3.16), we can choose $R > 1$ such that

$$\sqrt{2}\|u(t)\|^{\frac{p-1}{2}}_{L^2\left(x\leq m(t)-R+1\right)}\sup_{t\in\mathbb{R}}\|u(t)\|^{\frac{p-1}{2}}_{H^1} \leq \frac{p+1}{4p}\eta.$$

Taking into account (3.36), this leads eventually to the following estimate

$$\frac{2p}{p+1}\left|\int_{\mathbb{R}} u^{p+1}(t,x)\varphi'(\tilde{x})\ dx\right| \leq -\frac{\eta}{2}\int_{\mathbb{R}} u^2(t,x)\varphi'(\tilde{x})\ dx + C_0e^{-\kappa\left(-x_0-R+f(t)-f(t_0)\right)}, \tag{3.37}$$

where $C_0 := \frac{2p}{p+1}C$ is independent of x_0, t_0, and t. Gathering (3.31), (4.17), and (4.21) in (4.15) we deduce finally

$$\frac{dI_{(t_0,x_0)}}{dt}(t) \geq -3\int_{\mathbb{R}} u_x^2(t,x)\varphi'(\tilde{x})\ dx - C_0e^{-\kappa\left(-x_0-R+f(t)-f(t_0)\right)}.$$

Thus Lemma 3.2 is established. □

As a consequence of the preceding lemma,

$$\exists\ C_1 > 0,\ \forall\ x_0 \in \mathbb{R},\ \forall\ t \geq t_0, \qquad I_{(t_0,x_0)}(t_0) \leq I_{(t_0,x_0)}(t) + C_1e^{\kappa x_0}, \tag{3.38}$$

with C_1 independent of the parameters x_0 and t_0. Next, we claim the following:

Lemma 3.3. *For fixed* $x_0 \in \mathbb{R}$ *and* $t_0 \in J$, $I_{(t_0,x_0)}(t) \to 0$ *as* $t \to +\infty$.

Proof. To show this lemma, we just repeat the arguments given by Laurent and Martel [57, paragraph 2.1, Step 2]. Let ε be a positive real number. By Claim 3.14 and by (3.19), there exists $\tilde{R} > 0$ such that

$$\int_{x<\tilde{m}(t)-\tilde{R}} u^2(t,x)\, dx \leq \frac{\varepsilon}{2}.$$

Since $0 \leq \varphi \leq 1$, this enables us to see that

$$\int_{x<-\tilde{R}} u^2\left(t, x+\tilde{m}(t)\right) \varphi\big(x - x_0 + f(t) - f(t_0)\big)\, dx \leq \int_{x<\tilde{m}(t)-\tilde{R}} u^2(t,x)\, dx \leq \frac{\varepsilon}{2}. \tag{3.39}$$

Now, recall that φ is decreasing so that

$$\begin{aligned}\int_{x\geq-\tilde{R}} & u^2(t, x+\tilde{m}(t))\varphi\big(x - x_0 + f(t) - f(t_0)\big)\, dx \\ &\leq \varphi\big(-\tilde{R} - x_0 + f(t) - f(t_0)\big)\|u(t)\|_{L^2}^2 \\ &\leq \overline{C}\varphi\big(-\tilde{R} - x_0 + f(t) - f(t_0)\big),\end{aligned} \tag{3.40}$$

with $\overline{C} = \|u(t)\|_{L^2}^2$ for all $t \in J$. Moreover, since $f(t) \to +\infty$ as $t \to +\infty$ and $\varphi(x) \to 0$ as $x \to +\infty$, there exists $T \in \mathbb{R}$ such that for all $t \geq T$,

$$\overline{C}\varphi\big(-\tilde{R} - x_0 + f(t) - f(t_0)\big) \leq \frac{\varepsilon}{2}.$$

Then, for all $t \geq T$,

$$I_{(t_0,x_0)}(t) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence, we have finished proving Lemma 3.3. □

Due to (4.22) and Lemma 3.3, we obtain

$$\forall\, t_0 \in J,\ \forall\, x_0 \in \mathbb{R}, \qquad I_{(t_0,x_0)}(t_0) \leq C_1 e^{\kappa x_0}. \tag{3.41}$$

Thus, (4.13) leads to: for all $t \in J$,

$$\int_{x_0\leq x<0} u^2(t, x+\tilde{m}(t))e^{-\kappa x}\, dx \leq \frac{C_1}{\lambda_1}.$$

Thus letting x_0 tend to $-\infty$, we deduce from (4.12) that

$$\int_{x<\tilde{m}(t)} u^2(t,x)e^{\kappa(\tilde{m}(t)-x)}\, dx \leq \frac{C_1}{\lambda_1}.$$

Step 3: Proof of (3.23) for $k \geq 1$

Starting from the fact that for all $t \geq t_0$ and for all $x_0 < 0$

$$I_{(t_0,x_0)}(t_0) - I_{(t_0,x_0)}(t) \leq \frac{C_1}{\lambda_1}e^{\kappa x_0} + 3\int_{t_0}^{t}\int_{\mathbb{R}} u_x^2(s, x+\tilde{m}(s))\varphi'(x - x_0 + f(s) - f(t_0))\, dx\, ds$$

and arguing like Laurent and Martel [57, paragraph 2.1 Step 3], one can show

$$\int_{t_0}^{t_0+1} \int_{\mathbb{R}} u_x^2\,(s, x + \tilde{m}(s))\, e^{\kappa x}\; dx\; ds \le C,$$

where C is independent of x_0 and t_0.

Now, one proves by induction on $k \in \mathbb{N}$ the existence of $C_k \ge 0$ such that for all $t \in J$, $\|u(t)\|_{H^k} \le C_k$ and

$$\int_{\mathbb{R}} \left(\partial_x^k u\right)^2 (t, x + \tilde{m}(t))\, e^{\kappa x}\; dx + \int_t^{t+1} \int_{\mathbb{R}} \left(\partial_x^k u\right)^2 (s, x + \tilde{m}(s))\, e^{\kappa x}\; dx\; ds \le C_k. \tag{3.42}$$

In particular, estimates (3.23) are then performed. Moreover, we deduce from the equation satisfied by u that the partial derivatives with respect to x and t of all order exist and are continuous, thus $u \in \mathscr{C}^\infty(J \times \mathbb{R})$.

For simplification purposes, we will not explicit the proof of (3.42) and refer instead to [57, paragraph 2.3 and paragraph 2.2 Step 2]; the induction argument works since we assume $b'(t) \le \beta$ in (3.18), which implies that $\tilde{m}'$ is bounded on J. □

3.3 Proof of Theorem 3.4 and Corollary 3.1

We split the proof into four steps. The first three steps are common to both theorems and are valid under the hypotheses of Theorem 3.4, whereas the last one is specific to the proof of Corollary 3.1 where exponential decay properties are established and for which the stronger non dispersion assumption (3.10) is required.
Consider u which satisfies the assumptions of Theorem 3.4.

Step 1: Asymptotic stability in the energy space

The following asymptotic stability result in the energy space is to be considered as a crucial tool for the proof. In particular, we point out the importance of assumption (3.8) in Theorem 3.4.

Theorem 3.15 (Martel, Merle and Tsai [80]; Martel and Merle [72]). *Fix $0 < c_1^0 < \cdots < c_N^0$. For all $\beta > 0$, there exist $L_0 > 0$ and $\alpha_0 = \alpha_0(\beta) > 0$ such that if $u \in \mathscr{C}([0, +\infty), H^1(\mathbb{R}))$ is a solution of (gKdV) satisfying*

$$\forall\, t \ge 0, \qquad \inf_{\substack{r_i \in \mathbb{R} \\ r_{i+1} - r_i > L_0}} \left\| u(t) - \sum_{i=1}^N Q_{c_i^0}(\cdot - r_i) \right\|_{H^1} < C\left(\alpha_0 + e^{-\gamma t}\right) \tag{3.43}$$

for some positive constants C and γ, then the following holds.

1. *(Asymptotic stability in the energy space) There exist $\mathscr{C}^1$ functions $t \mapsto c_i(t) \in \mathbb{R}_+^*$, $t \mapsto \rho_i(t) \in \mathbb{R}$ for $i \in \{1, \dots, N\}$ such that*

$$\lim_{t \to +\infty} \left\| u(t) - \sum_{i=1}^N Q_{c_i(t)}(\cdot - \rho_i(t)) \right\|_{H^1\left(x > \beta t\right)} = 0. \tag{3.44}$$

2. *(Convergence of the scaling parameter) There exists* $c_i^+ \in \mathbb{R}_+^*$ *such that* $\lim_{t\to+\infty} c_i(t) = c_i^+$.

Set $\delta := \frac{1}{2}\min\left\{c_1, \min_{i\in\{1,\dots,N-1\}}\{c_{i+1}-c_i\}\right\}$.
Using Theorem 3.15 and adapting the classical modulation argument set up in the proof by Martel and Merle [72, section 5], we have the following. For all $i \in \{1,\dots,N\}$, there exist $c_i : [0,+\infty) \to \mathbb{R}_+^*$, $\rho_i : [0,+\infty) \to \mathbb{R}$ of class $\mathscr{C}^1$ such that, defining

$$\epsilon : (t,x) \mapsto u(t,x) - \sum_{i=1}^{N} Q_{c_i(t)}(x-\rho_i(t)), \tag{3.45}$$

and for α small enough in Theorems 3.4 and 3.1, we have

1. the conclusion of Theorem 3.15, that is

$$\lim_{t\to+\infty} \|\epsilon(t)\|_{H^1\left(x>\frac{c_1}{A}t\right)} = 0, \tag{3.46}$$

with $A > 3$ such that

$$\frac{c_1}{A} < \rho - \sigma, \tag{3.47}$$

(take $\sigma = 0$ for Theorem 3.4) and

$$\forall\, i \in \{1,\dots,N\},\ \exists\, c_i^+ \in \mathbb{R}_+^*, \qquad \lim_{t\to+\infty} c_i(t) = c_i^+; \tag{3.48}$$

2. control on the modulation parameters [80, proof of Lemma 1]: more precisely, there exists $K > 0$ such that for all t large enough,

$$\forall\, i \in \{1,\dots,N-1\}, \quad \rho_{i+1}(t) - \rho_i(t) \geq \delta t, \tag{3.49}$$

and for all $i \in \{1,\dots,N\}$,

$$|c_i(t) - c_i| + \|\epsilon(t)\|_{H^1} \leq \frac{\delta}{K+1}, \tag{3.50}$$

$$|\rho_i'(t) - c_i(t)| \leq K\left(\int_{\mathbb{R}} \epsilon^2(t,x) e^{-\sqrt{c_1}|x-\rho_i(t)|}\, dx\right)^{\frac{1}{2}}. \tag{3.51}$$

Remark 3.16. The preceding choices of A, K, and of the functions $t \mapsto c_i(t)$ as defined before are possible, provided α is sufficiently small.
Note also that estimate (3.50) and assertion (3.48) guarantee that $0 < c_1^+ < \cdots < c_N^+$ (due to the choice of δ).

Step 2: Convergence of $u(t) - \sum_{i=1}^N Q_{c_i^+}(\cdot - \rho_i(t))$ *as* $t \to +\infty$

Lemma 3.4. *We have*

$$\left\| u(t) - \sum_{i=1}^{N} Q_{c_i^+}(\cdot - \rho_i(t)) \right\|_{H^1} \to 0, \qquad \text{as } t \to +\infty.$$

This lemma follows immediately from Claim 3.17 and Claim 3.18 below. We begin with this first observation.

Claim 3.17. *We have*

$$\lim_{t\to+\infty}\left\|u(t)-\sum_{i=1}^{N}Q_{c_i^+}(\cdot-\rho_i(t))\right\|_{H^1\left(x>\frac{c_1}{A}t\right)}=0. \tag{3.52}$$

Let us justify this fact.
Using the triangular inequality and taking into account (3.46), it suffices in fact to see that for all $i\in\{1,\dots,N\}$,

$$\lim_{t\to+\infty}\left\|Q_{c_i(t)}(\cdot-\rho_i(t))-Q_{c_i^+}(\cdot-\rho_i(t))\right\|_{H^1\left(x>\frac{c_1}{A}t\right)}=0. \tag{3.53}$$

But the quantity $\left\|Q_{c_i(t)}(\cdot-\rho_i(t))-Q_{c_i^+}(\cdot-\rho_i(t))\right\|_{H^1\left(x>\frac{c_1}{A}t\right)}$ is bounded by $\|Q_{c_i(t)}-Q_{c_i^+}\|_{H^1}$ which tends to 0 as t tends to $+\infty$. We recall indeed that the map

$$\mathbb{R}_+^*\to H^1(\mathbb{R}),\quad c\mapsto Q_c$$

is continuous by application of Lebesgue's dominated convergence theorem. Hence (3.53) holds and Claim 3.17 is proved.

Due to the assumption of non dispersion made in Theorem 3.4, we claim moreover:

Claim 3.18. *We have*

$$\lim_{t\to+\infty}\left\|u(t)-\sum_{i=1}^{N}Q_{c_i^+}(\cdot-\rho_i(t))\right\|_{H^1\left(x\le\frac{c_1}{A}t\right)}=0. \tag{3.54}$$

In what follows, we prove actually that each quantity $\|u(t)\|_{H^1\left(x\le\frac{c_1}{A}t\right)}$ and $\|Q_{c_i^+}(\cdot-\rho_i(t))\|_{H^1\left(x\le\frac{c_1}{A}t\right)}$ for $i\in\{1,\dots,N\}$ tends to 0 as t tends to $+\infty$.

1. Proof of $\|u(t)\|_{H^1\left(x\le\frac{c_1}{A}t\right)}\underset{t\to+\infty}{\longrightarrow}0$.
   Let $\varepsilon>0$. By (3.6) or (3.10), there exists $R_\varepsilon>0$ such that for all $R\ge R_\varepsilon$,

   $$\forall\, t\ge 0,\qquad \int_{x<(\rho-\sigma)t-R}u^2(t,x)\,dx\le\frac{\varepsilon}{2}.$$

   Now, by means of Proposition 3.11, there exist $\kappa>0$ and $K_1>0$ such that for all $t\ge 0$,

   $$\forall\, x\le(\rho-\sigma)t,\qquad |u_x(t,x)|\le K_1e^{-\kappa|x-(\rho-\sigma)t|}.$$

   Pick $R\ge R_\epsilon$ such that $K_1^2e^{-\kappa R}\displaystyle\int_{\mathbb{R}}e^{-\kappa|x|}\,dx\le\frac{\varepsilon}{2}$.
   For t large enough, $\frac{c_1}{A}t<(\rho-\sigma)t-R$ due to (3.47), and therefore

   $$\int_{x\le\frac{c_1}{A}t}u^2(t,x)\,dx\le\frac{\varepsilon}{2},$$

and

$$\int_{x\leq\frac{c_1}{A}t} u_x^2(t,x)\,dx \leq \int_{x\leq(\rho-\sigma)t-R} u_x^2(t,x)\,dx \leq K_1\int_{x\leq(\rho-\sigma)t-R} e^{-\kappa R}|u_x(t,x)|\,dx$$
$$\leq K_1^2 e^{-\kappa R}\int_{x\leq(\rho-\sigma)t-R} e^{-\kappa|x-(\rho-\sigma)t|}\,dx$$
$$\leq K_1^2 e^{-\kappa R}\int_{\mathbb{R}} e^{-\kappa|x-(\rho-\sigma)t|}\,dx \leq \frac{\varepsilon}{2}.$$

As a consequence, for t large enough, $\|u(t)\|^2_{H^1\left(x\leq\frac{c_1}{A}t\right)} \leq \varepsilon$.

2. Proof of $\|Q_{c_i^+}(\cdot-\rho_i(t))\|_{H^1\left(x\leq\frac{c_1}{A}t\right)} \underset{t\to+\infty}{\longrightarrow} 0$.
   Notice first that, recalling (3.50) and (3.51), we have for t large enough
   $$|\rho_i'(t)-c_i| \leq |\rho_i'(t)-c_i(t)| + |c_i(t)-c_i| \leq K\|\epsilon(t)\|_{L^2} + |c_i(t)-c_i|$$
   $$\leq (K+1)\big(\|\epsilon(t)\|_{L^2} + |c_i(t)-c_i|\big) \leq \frac{c_1}{2}.$$
   In particular, for t large enough,
   $$\rho_i'(t) \geq \frac{c_1}{2}. \tag{3.55}$$
   By integration of the preceding inequality, we deduce that for large values of t, $\rho_i(t)\geq\frac{c_1}{3}t$. Thus, for these values,
   $$\rho_i(t) - \frac{c_1}{A}t \geq c_1\left(\frac{1}{3}-\frac{1}{A}\right)t, \tag{3.56}$$
   with $\frac{1}{3}-\frac{1}{A} > 0$.
   Due to the exponential decay property of the integrable functions $Q_{c_i^+}$ and $Q'_{c_i^+}$, we deduce then from (3.56) that $\|Q_{c_i^+}(\cdot-\rho_i(t))\|_{H^1\left(x\leq\frac{c_1}{A}t\right)} \underset{t\to+\infty}{\longrightarrow} 0$.

Now, it follows from Claim 3.17 and Claim 3.18 that

$$\lim_{t\to+\infty}\left\|u(t)-\sum_{i=1}^N Q_{c_i^+}(\cdot-\rho_i(t))\right\|_{H^1} = 0. \tag{3.57}$$

Step 3: Refinement of (3.57)

Proposition 3.19 (Improvement of the H^1-convergence for asymptotic N-soliton like solutions)**.** *Let $u\in\mathscr{C}(\mathbb{R},H^1(\mathbb{R}))$ be a solution of (gKdV) and let $0<c_1<\cdots<c_N$. Assume the existence of $T_0\geq 0$, $\delta_0>0$, and N functions $x_1,\ldots,x_N:\mathbb{R}\to\mathbb{R}$ of class $\mathscr{C}^1$ satisfying for all $t\geq T_0$,*

$$\forall\, i=1,\ldots,N-1,\quad x_{i+1}(t)-x_i(t)\geq\delta_0 t \qquad \text{and} \qquad \forall\, i=1,\ldots,N,\quad x_i'(t)\geq\delta_0, \tag{3.58}$$

and such that

$$\lim_{t\to+\infty}\left\|u(t)-\sum_{i=1}^N Q_{c_i}(\cdot-x_i(t))\right\|_{H^1} = 0. \tag{3.59}$$

Then there exist $C>0$ and $y_1,\ldots,y_N\in\mathbb{R}$ such that

$$\forall\, t\geq 0,\qquad \left\|u(t)-\sum_{i=1}^N Q_{c_i}(\cdot-c_it-y_i)\right\|_{H^1} \leq Ce^{-\frac{1}{8}\delta_0^{\frac{3}{2}}t}. \tag{3.60}$$

It was first observed by Martel [63, Proposition 4] that multi-solitons in the sense of Definition 4.1 do actually converge exponentially fast to their profile: this was a key to proving uniqueness of multi-solitons, in the L^2-subcritical case. In the above Proposition 3.19 we further refine this observation, by noticing that the conclusion still holds even if one gives some freedom to the center of mass of the soliton $x_i(t)$ (instead of (3.58), the assumption in [63, Proposition 4] was $x_i(t) = c_i t + y_i$).

The proof of Proposition 3.19 follows the lines of [63] and is postponed to the Appendix; we go on assuming it holds.

Given (3.49), (3.55), and (3.57), we just have to apply the previous proposition (with x_i replaced by ρ_i and δ_0 by δ defined in Step 1) to conclude that u is a multi-soliton. In other words, there exist $x_1^+, \dots, x_N^+ \in \mathbb{R}$ such that

$$\forall\, t \geq 0, \qquad \left\| u(t) - \sum_{i=1}^{N} Q_{c_i^+}(\cdot - c_i^+ t - x_i^+) \right\|_{H^1} \leq C e^{-\frac{1}{8}\delta^{\frac{3}{2}} t}. \tag{3.61}$$

We recall then from Martel [63, proof of Proposition 5] that for all $s \in \mathbb{N}^*$, $u \in \mathscr{C}([0,+\infty), H^s(\mathbb{R}))$ and there exists $\tilde{C}_s \geq 0$ such that

$$\forall\, t \geq 0, \qquad \left\| u(t) - \sum_{i=1}^{N} Q_{c_i^+}(\cdot - c_i^+ t - x_i^+) \right\|_{H^s} \leq \tilde{C}_s e^{-\frac{1}{32}\delta^{\frac{3}{2}} t}. \tag{3.62}$$

This concludes the proof of Theorem 3.4.

Step 4: Proof of smoothness and exponential decay of u

Apply Proposition 3.12 with $a(t) := (\rho - \sigma)t$ and $b(t) := (\rho + \sigma)t$ to obtain $u \in \mathscr{C}^\infty([0,+\infty) \times \mathbb{R})$ and for each $t \geq 0$,

$$\forall\, x \leq (\rho - \sigma)t, \qquad \left| \partial_x^s u(t,x) \right| \leq K_s e^{-\gamma |x - (\rho-\sigma)t|},$$

where $K_s, \gamma > 0$ are independent of t and x.
Under the global non dispersion assumption of Corollary 3.1, which we take as granted from now on, we have also

$$\forall\, x \geq (\rho + \sigma)t, \qquad \left| \partial_x^s u(t,x) \right| \leq K_s e^{-\gamma (x - (\rho+\sigma)t)}.$$

(See Remark 3.13.)

At this stage and as we explain just below, the desired exponential decay estimate (3.11) follows from the strong property (3.5) or (3.62). We distinguish three cases, depending on the position of x with respect to $\pm 2(\rho+\sigma)t$; the moral being that (3.5) implies the expected pointwise estimate in each region $|x| \leq \zeta t$ (with an exponential decay rate depending on ζ) and even if it means taking ζ large enough and reducing the decay rate γ, one can propagate the control by $e^{-\gamma|x - c_N^+ t|}$ (respectively $e^{-\gamma|x - c_1^+ t|}$) to the region $x > (\rho + \sigma)t$ (respectively $x < (\rho - \sigma)t$).

Let $t \geq 0$ and $\tilde{\theta} := \frac{1}{32}\delta^{\frac{3}{2}}$, for each $s \in \mathbb{N}$, there exists $\tilde{K}_s > 0$ such that

$$\left\| \partial_x^s \left(u(t) - \sum_{i=1}^{N} Q_{c_i^+}(\cdot - c_i^+ t - x_i^+) \right) \right\|_{L^\infty} \leq \tilde{K}_s e^{-\tilde{\theta} t}.$$

Case 1: $|x| \leq 2(\rho+\sigma)t$. We have

$$|x - c_N^+ t| \leq (2(\rho+\sigma) + c_N^+)t \quad \text{that is,} \quad \frac{\tilde{\theta}}{2(\rho+\sigma)+c_N^+}|x - c_N^+ t| \leq \tilde{\theta} t,$$

and thus

$$\left\| \partial_x^s \left(u(t) - \sum_{i=1}^N Q_{c_i^+}(\cdot - c_i^+ t - x_i^+) \right) \right\|_{L^\infty} \leq \tilde{K}_s e^{-\frac{\tilde{\theta}}{2(\rho+\sigma)+c_N^+}|x - c_N^+ t|}. \tag{3.63}$$

Consequently for $t \geq 0$ and $|x| \leq 2(\rho+\sigma)t$, using the triangular inequality and the exponential decay of $\partial_x^s Q_{c_i^+}$, we obtain

$$\begin{aligned} |\partial_x^s u(t,x)| &\leq \sum_{i=1}^N \left| \partial_x^s Q_{c_i^+}(x - c_i^+ t - x_i^+) \right| + \left\| \partial_x^s \left(u(t) - \sum_{i=1}^N Q_{c_i^+}(\cdot - c_i^+ t - x_i^+) \right) \right\|_{L^\infty} \\ &\leq \tilde{\tilde{K}}_s \sum_{i=1}^N e^{-\tilde{\gamma}|x - c_i^+ t|}, \end{aligned} \tag{3.64}$$

where $\tilde{\gamma} := \min\left\{ \sqrt{c_1^+}, \frac{\tilde{\theta}}{2(\rho+\sigma)+c_N^+} \right\}$.

Case 2: $x \geq 2(\rho+\sigma)t$. Let us rewrite this as $x - (\rho+\sigma)t \geq \frac{1}{2}x$. In particular

$$x - (\rho+\sigma)t \geq \frac{1}{2}\left(x - c_N^+ t\right)$$

so that for $x \geq 2(\rho+\sigma)t$,

$$\left| \partial_x^s u(t,x) \right| \leq K_s e^{-\gamma(x - (\rho+\sigma)t)} \leq K_s e^{-\frac{\gamma}{2}(x - c_N^+ t)}. \tag{3.65}$$

Case 3: $x \leq -2(\rho+\sigma)t$. Arguing similarly as before, we have then

$$\left| \partial_x^s u(t,x) \right| \leq K_s e^{-\gamma((\rho-\sigma)t - x)} \leq K_s e^{-\frac{\gamma}{L}(c_1^+ t - x)}, \tag{3.66}$$

for $L > 2$ chosen such that $\frac{L(\rho-\sigma)-c_1^+}{L-1} > -2(\rho+\sigma)$.
Set finally $\theta := \min\left\{ \frac{\gamma}{L}, \tilde{\gamma} \right\}$ to obtain (3.11) in Corollary 3.1.

3.4 The integrable cases: proofs of Theorems 3.7 and 3.9

3.4.1 Non dispersive solutions of the Korteweg-de Vries equation

The strategy to prove Theorem 3.7 takes inspiration in [57, Proof of Theorem 2]. We use the following result of Eckhaus and Schuur [28, Section 5], which is also a consequence of a generalized version by Schuur [101, Chapter 2, Theorem 7.1 and (7.23)].

Theorem 3.20 (Eckhaus and Schuur [28]; Schuur [101])**.** *Let $p = 2$ and $u_0 \in \mathscr{C}^4(\mathbb{R})$ be such that for some $C_0 > 0$, for all $k = 0, \dots, 4$, and for all $x \in \mathbb{R}$,*

$$\left| \frac{\partial^k u_0(x)}{\partial x^k} \right| \leq C_0 |x|^{-11}. \tag{3.67}$$

Let u be the corresponding global solution of (KdV).

Then there exists a solution u_d which is a multi-soliton or zero such that for all $\beta > 0$, there exists $K \geq 0$ such that for all $t > 0$

$$\|u(t) - u_d(t)\|_{L^\infty(x>\beta t)} + \|u(t) - u_d(t)\|_{L^2(x>\beta t)} \leq K t^{-\frac{1}{3}}. \tag{3.68}$$

Proof of Theorem 3.7. Due to the assumption on u_0 in Theorem 3.7, we can apply Theorem 3.20 and we obtain a solution u_d of (KdV) as above which fulfills (3.68). We claim first that u_d is not the trivial solution. Otherwise, with $\beta := \frac{\rho}{2} > 0$, we would have $\|u(t)\|_{L^2(x>\beta t)} = \mathrm{O}(t^{-\frac{1}{3}})$ as t tends to $+\infty$. On the other hand, by the non dispersion assumption and namely by Proposition 3.11,

$$\forall\, x \leq \beta t, \qquad |u(t,x)| \leq C e^{-\gamma|x-\rho t|}, \tag{3.69}$$

so that

$$\|u(t)\|_{L^2(x\leq\beta t)} \leq C e^{-\gamma\beta t}. \tag{3.70}$$

Then, we would obtain that

$$\|u(t)\|_{L^2} = \|u(t)\|_{L^2(x\leq\beta t)} + \|u(t)\|_{L^2(x>\beta t)} \longrightarrow 0 \qquad \text{as } t \longrightarrow +\infty,$$

hence conclude that $\|u_0\|_{L^2} = 0$ by the mass conservation law. This contradicts our assumption in Theorem 3.7.

Thus there exist $N \geq 1$, $0 < c_1 < \cdots < c_N$, $x_1^+, \ldots, x_N^+ \in \mathbb{R}$, and a possibly smaller $\gamma > 0$ such that

$$\left\| u_d(t) - \sum_{i=1}^{N} R_{c_i,x_i^+}(t) \right\|_{H^1} = \mathrm{O}\left(e^{-\gamma t}\right), \quad \text{as } t \longrightarrow +\infty. \tag{3.71}$$

Claim 3.21. *We have*

$$\left\| u(t) - \sum_{i=1}^{N} R_{c_i,x_i^+}(t) \right\|_{L^2} = \mathrm{O}\left(t^{-\frac{1}{3}}\right), \qquad \text{as } t \longrightarrow +\infty.$$

Proof of Claim 3.21. Consider $\beta \in (0, \min\{c_1, \rho\})$ so that (by the non dispersion assumption and the sech-shaped profiles of the solitons R_{c_i,x_i^+})

$$\|u(t)\|_{L^2(x\leq\beta t)} + \sum_{i=1}^{N} \left\| R_{c_i,x_i^+}(t) \right\|_{L^2(x\leq\beta t)} = \mathrm{O}\left(e^{-\gamma t}\right)$$

even if it means reducing $\gamma > 0$. We perform then

$$\begin{aligned}
\|u(t) - u_d(t)\|_{L^2} &= \|u(t) - u_d(t)\|_{L^2(x\leq\beta t)} + \|u(t) - u_d(t)\|_{L^2(x>\beta t)} \\
&\leq \|u(t)\|_{L^2(x\leq\beta t)} + \|u_d(t)\|_{L^2(x\leq\beta t)} + \mathrm{O}\left(t^{-\frac{1}{3}}\right) \\
&\leq \|u_d(t)\|_{L^2(x\leq\beta t)} + \mathrm{O}\left(t^{-\frac{1}{3}} + e^{-\gamma t}\right) \\
&\leq \left\| u_d(t) - \sum_{i=1}^{N} R_{c_i,x_i^+}(t) \right\|_{H^1} + \sum_{i=1}^{N} \left\| R_{c_i,x_i^+}(t) \right\|_{L^2(x\leq\beta t)} + \mathrm{O}\left(t^{-\frac{1}{3}}\right) \\
&= \mathrm{O}\left(t^{-\frac{1}{3}}\right),
\end{aligned}$$

by the embeddings $H^1(\mathbb{R}) \hookrightarrow H^1(x \leq \beta t) \hookrightarrow L^2(x \leq \beta t)$ and by (3.71). By means of the triangular inequality and once again (3.71), we deduce the expected estimate in Claim 3.21. □

We are now able to finish the proof of Theorem 3.7. Indeed, let us make the following key observation.

Claim 3.22. *The solution u belongs to $L^\infty([0,+\infty), H^2(\mathbb{R}))$.*

Proof of Claim 3.22. This is an immediate consequence of the following conservation law for the KdV equation

$$\frac{d}{dt}\int_{\mathbb{R}}\left\{\left(\partial_x^2 u\right)^2 - \frac{10}{3}\left(\partial_x u\right)^2 u + \frac{5}{9}u^4\right\}(t,x)\,dx = 0, \tag{3.72}$$

of the Sobolev embedding $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$, and from the fact that u belongs to $L^\infty(\mathbb{R}, H^1(\mathbb{R}))$. □

As a consequence of Claim 3.22, $v := u - \sum_{i=1}^N R_{c_i,x_i^+}$ belongs also to $L^\infty([T_1,+\infty), H^2(\mathbb{R}))$. Then, integrating by parts and using the Cauchy-Schwarz inequality and Claim 3.21, we infer that

$$\int_{\mathbb{R}}\left(\partial_x v\right)^2(t)\,dx = -\int_{\mathbb{R}} v(t)\partial_x^2 v(t)\,dx \le \|v(t)\|_{L^2}\|v(t)\|_{H^2} \le Ct^{-\frac{1}{3}},$$

from which it results that

$$\left\|u(t) - \sum_{i=1}^N R_{c_i,x_i^+}(t)\right\|_{H^1} \to 0, \qquad \text{as } t \to +\infty. \tag{3.73}$$

Hence u is a multi-soliton in $+\infty$. By means of the well-known theory concerning multi-solitons of the KdV equation (see for instance Miura [89]), we deduce that u is also a multi-soliton in $-\infty$. This ends the proof of Theorem 3.7. □

3.4.2 Non dispersive solutions of the modified Korteweg-de Vries equation

Theorem 3.9 is obtained by using the same strategy as that developed in the previous subsection. Thus we will only sketch its proof.

As for the KdV case, we apply first the following decomposition result, obtained from [101, Chapter 5, Theorem 5.1] and from [8, Theorem 1.10] where a more precise version can be found.

Theorem 3.23 (Schuur [101], Chen and Liu [8])**.** *Let $p = 3$ and $u_0 \in \mathscr{C}^4(\mathbb{R})$ be such that for some $C_0 > 0$, for all $k = 0, \dots, 4$, and for all $x \in \mathbb{R}$,*

$$\left|\frac{\partial^k u_0(x)}{\partial x^k}\right| \le C_0|x|^{-11}, \tag{3.74}$$

and be generic as in Theorem 3.9, with scattering data (3.13)*. Let u be the corresponding global solution of (mKdV).*

Then there exist signs $\epsilon_i = \pm 1$, $i = 1, \dots, N_1$, and parameters $x_{0,i}$, $i = 1, \dots, N_1$, and $x_{1,j}$, $x_{2,j}$, $j = 1, \dots, N_2$, such that for all $v_+ > 0$ and $v_- < 0$, there exists $K \ge 0$ such that for all $t > 0$, denoting

$$P(t) := \sum_{i=1}^{N_1} \epsilon_i R_{2c_i,x_{0,i}}(t) + \sum_{j=1}^{N_2} B_{\sqrt{2}\alpha_j,\sqrt{2}\beta_j,x_{1,j},x_{2,j}}(t),$$

we have

$$\|u(t) - P(t)\|_{L^\infty(x>v_+t)} + \|u(t) - P(t)\|_{L^2(x>v_+t)} \leq Kt^{-\frac{1}{3}}, \tag{3.75}$$

and

$$\|u(t) - P(t)\|_{L^\infty(x<v_-t)} \leq Kt^{-\frac{1}{2}}, \tag{3.76}$$

Then, the non dispersion assumption (3.6) in Theorem 3.9 shows that $N_1 + N_2 \geq 1$. Since the profiles of the breathers are sech-shaped, due to (3.6) and (3.76), we deduce that the breathers have positive (envelope) velocities.

Now, proceeding as in the proof of Claim 3.21, we obtain in fact that

$$\|u(t) - P(t)\|_{L^2} = \mathrm{O}\left(t^{-\frac{1}{3}}\right).$$

Moreover, (mKdV) admits conservation laws of orders 2, 3, and 4 in the spirit of (3.72), which shows that u belongs to $L^\infty([0,+\infty), H^4(\mathbb{R}))$. Proceeding similarly to subsection 3.4.1, we obtain that

$$\|u(t) - P(t)\|_{H^2} \to 0, \qquad \text{as } t \to +\infty.$$

Finally, by the uniqueness and smoothness results and the estimates in higher Sobolev spaces proven by Semenov [102] as far as multi-breathers are concerned, we deduce that u belongs to $\mathscr{C}([0,+\infty), H^s(\mathbb{R}))$ and that there exist $\gamma > 0$ and positive constants C_s such that for all $s \in \mathbb{N}$,

$$\|u(t) - P(t)\|_{H^s} \leq C_s e^{-\gamma t}, \qquad \text{as } t \to +\infty.$$

This finishes proving Theorem 3.9.

3.5 Appendix

3.5.1 Proof of Proposition 3.19

The proof follows the same lines as that of Proposition 3 and paragraph 3.2 in Martel [63] for the L^2-subcritical and critical cases, and that of Lemma 4.1 in Combet [11] for the supercritical case. For the sake of simplicity and for the reader's convenience, we present here the essential ideas and also the changes in the L^2-subcritical case only.

Remark 3.24. We mention that in the L^2-critical and supercritical cases, the proof is basically changed in terms of the coercivity property we use to control the modulation function ϵ defined below in Step 1. The monotonicity properties of local mass and energy obtained in Step 2 are still valid in these cases.

Concerning the critical case, the idea is to modulate the scaling parameter in addition to the translation parameter so as to ensure a second orthogonality condition satisfied by ϵ, namely $\int_{\mathbb{R}} \epsilon(t)\tilde{R}_j(t)^3 \, dx = 0$, and then to apply a localized version of the coercivity property available in this case, which leads to:

$$\exists\, \lambda_0 > 0,\ \forall\, t, \qquad \|\epsilon(t)\|_{H^1}^2 \leq \lambda_0 \mathcal{H}(t), \tag{3.77}$$

with $\mathcal{H}$ defined in Step 3.

In the supercritical case, it is known from Pego and Weinstein [96] that, considering the standard linearized operator L on $H^1(\mathbb{R})$ defined by $Lv := -\partial_x^2 v + v - pQ^{p-1}v$, the composed operator $L\partial_x$

has two eigenfunctions Z^+ and Z^- related by $Z^-(x) = Z^+(-x)$, which decay exponentially, and such that $L\partial_x Z^\pm = \pm e_0 Z^\pm$ for some $e_0 > 0$. In this case, we only have to make modifications in Step 3 by using this time

$$\exists\, \lambda_0 > 0,\ \forall\, t, \qquad \|\epsilon(t)\|_{H^1}^2 \leq \lambda_0 \mathcal{H}(t) + \frac{1}{\lambda_0}\sum_{i,\pm}\left(\int_{\mathbb{R}} \epsilon(t)\tilde{Z}_i^\pm(t)\ dx\right)^2, \tag{3.78}$$

where $\tilde{Z}_i^\pm(t) := Z_i^\pm(\cdot - x_i(t) - y_i(t))$ and $Z_i^\pm(x) := c_i^{-\frac{1}{2}} Z^\pm\left(c_i^{\frac{1}{2}} x\right)$.

(The functions y_i are defined in Lemma 3.5 below.) The control of $\int_{\mathbb{R}} \epsilon(t)\tilde{Z}_i^\pm(t)\ dx$ by a function of t which decreases with exponential speed follows the strategy of Combet (for full details, see [11, paragraph 4.1 Step 4]).

Step 1: Set up of a modulation argument

Set $\nu := \min\{c_1, \delta_0\}$. We claim the following

Lemma 3.5. *There exist $T \geq 0$ and $\alpha_1 \in (0,1]$ such that for all $\tilde{\alpha} \leq \alpha_1$, the following holds. There exist unique $\mathscr{C}^1$ functions $y_i : [T, +\infty) \to \mathbb{R}$ such that defining*

$$\epsilon := u - \sum_{i=1}^N \tilde{R}_i, \tag{3.79}$$

where $\tilde{R}_i(t,x) := Q_{c_i}\big(x - x_i(t) - y_i(t)\big)$, we have for all $t \geq T$,

$$\forall\, i \in \{1, \dots, N\}, \qquad \int_{\mathbb{R}} \epsilon\big(\tilde{R}_i\big)_x(t)\ dx = 0. \tag{3.80}$$

In addition, there exists $K > 0$ such that for all $t \geq T$, for all $i \in \{1, \dots, N\}$,

$$\|\epsilon(t)\|_{H^1} + \sum_{i=1}^N |y_i(t)| \leq K\tilde{\alpha}, \tag{3.81}$$

$$|x_i'(t) + y_i'(t) - c_i| \leq K\left(\int_{\mathbb{R}} \epsilon^2(t) e^{-\sqrt{\nu}|x - x_i(t)|}\ dx\right)^{\frac{1}{2}} + K e^{-\frac{1}{4}\nu^{\frac{3}{2}} t}. \tag{3.82}$$

Proof. Recall that the proof of existence and uniqueness of the functions $y_i(t)$ is based on the implicit function theorem. We refer to [80, proof of Lemma 8] and also to [67, paragraph 2.3] for a complete proof in the case of one soliton. Moreover, estimate (3.82) which involves $\nu \leq c_1$ is obtained formally by writing the equation of ϵ, that is

$$\epsilon_t + \partial_x^3 \epsilon = \sum_{i=1}^N (x_i' + y_i' - c_i)(\tilde{R}_i)_x - \left(\left(\epsilon + \sum_{i=1}^N \tilde{R}_i\right)^p - \sum_{i=1}^N \tilde{R}_i^p\right)_x, \tag{3.83}$$

by multiplying it by $(\tilde{R}_i)_x$, and by using the following properties:

$$0 = \frac{d}{dt}\int_{\mathbb{R}} \epsilon(\tilde{R}_i)_x\ dx = \int_{\mathbb{R}} \partial_t \epsilon \partial_x \tilde{R}_i\ dx - (x_i' + y_i')\int_{\mathbb{R}} \epsilon \partial_x^2 \tilde{R}_i\ dx; \tag{3.84}$$

$\forall\, i \neq j$, $\forall\, t \geq T_2$,

$$|\tilde{R}_i(t,x)| + |\partial_x \tilde{R}_i(t,x)| \leq Ce^{-\sqrt{\nu}|x-x_i(t)|}; \tag{3.85}$$

$$\int_{\mathbb{R}} \left\{ \tilde{R}_i(t,x)\tilde{R}_j(t,x) + |\partial_x \tilde{R}_i(t,x)\partial_x \tilde{R}_j(t,x)| \right\} \, dx \leq Ce^{-\frac{\nu^{\frac{3}{2}}}{2}t}. \tag{3.86}$$

Note that (3.86) is a consequence of the decoupling assumption (3.58). We refer to [63] and the references therein for more details. □

Step 2: Monotonicity properties for localized mass and some modified energy of u

Let $\psi : x \mapsto \frac{2}{\pi}\mathrm{Arctan}\left(e^{-\frac{\sqrt{\nu}}{2}x}\right)$ be defined on $\mathbb{R}$ so that for all $x \in \mathbb{R}$,

$$\psi'(x) \leq 0, \qquad |\psi'(x)| \leq \frac{\sqrt{\nu}}{\pi}e^{-\frac{\sqrt{\nu}}{2}|x|}, \qquad |\psi^{(3)}(x)| \leq \frac{\nu}{4}|\psi'(x)|,$$

(we recall $\nu = \min\{c_1, \delta_0\}$). Then define on $\mathbb{R}^+ \times \mathbb{R}$:

$$\forall\, i \in \{1,\ldots,N-1\}, \quad \psi_i : (t,x) \mapsto \psi\left(x - \frac{x_i(t) + x_{i+1}(t)}{2}\right), \quad \psi_N : (t,x) \mapsto 1 \tag{3.87}$$

and also

$$\phi_1 := \psi_1, \quad \forall\, i \in \{2,\ldots,N-1\}, \quad \phi_i := \psi_i - \psi_{i-1}, \quad \phi_N := 1 - \psi_{N-1}. \tag{3.88}$$

Remark 3.25. Note that by definition and by (3.58), for $t > 0$, $\phi_i(t)$ takes values close to 1 in a neighborhood of $x_i(t)$ and takes values close to 0 around $x_j(t)$ for $j \neq i$.

Take $\kappa \in \left(0, \frac{c_1}{4}\right)$ and consider now for all $i \in \{1,\ldots,N-1\}$ the following quantities:

- (localized mass of u at the left of $\frac{x_i(t)+x_{i+1}(t)}{2}$)

$$\mathcal{M}_i(t) := \int_{\mathbb{R}} u^2(t,x)\psi_i(t,x) \, dx \tag{3.89}$$

- (modified localized energy of u at the left of $\frac{x_i(t)+x_{i+1}(t)}{2}$)

$$\tilde{\mathcal{E}}_i(t) := \int_{\mathbb{R}} \left(\frac{1}{2}u_x^2 - \frac{1}{p+1}u^{p+1} + \kappa u^2\right)\psi_i(t,x) \, dx. \tag{3.90}$$

Let also

$$\mathcal{M}_N(t) := \int_{\mathbb{R}} u^2(t,x)\psi_N(t,x) \, dx \tag{3.91}$$

(which is nothing but the mass of u) and

$$\tilde{\mathcal{E}}_N(t) := \int_{\mathbb{R}} \left(\frac{1}{2}u_x^2 - \frac{1}{p+1}u^{p+1} + \kappa u^2\right)\psi_N(t,x) \, dx \tag{3.92}$$

(which is a global quantity linked to the energy of u).

Remark 3.26. The reason why we have to choose κ small enough appears clearly in Step 3 (see Remark 3.27).

We claim now a monotonicity result on the preceding quantities.

Lemma 3.6. *There exist $T_1 \geq T_0$ and $K_1 \geq 0$ such that for all $t \geq T_1$ and for all $i \in \{1, \dots, N\}$,*

$$\frac{d\mathcal{M}_i}{dt}(t) \geq -K_1 e^{-\frac{\nu^{\frac{3}{2}}}{4}t} \qquad \text{and} \qquad \frac{d\tilde{\mathcal{E}}_i}{dt}(t) \geq -K_1 e^{-\frac{\nu^{\frac{3}{2}}}{4}t}. \tag{3.93}$$

Proof. First we observe that

$$\frac{d\mathcal{M}_i}{dt} = -\int_{\mathbb{R}} \left(3u_x^2 + \frac{x_i' + x_{i+1}'}{2}u^2 - \frac{2p}{p+1}u^{p+1}\right)\psi_i' \, dx + \int_{\mathbb{R}} u^2 \psi_i^{(3)} \, dx.$$

For all $\eta_0 > 0$, there exists $T_{\eta_0} \geq 0$ such that for all $t \geq T_{\eta_0}$,

$$\left\| u(t) - \sum_{i=1}^{N} Q_{c_i}(\cdot - x_i(t)) \right\|_{H^1} \leq \eta_0$$

and for all $R_0 > 0$, for each $(t,x) \in \mathbb{R}^+ \times \mathbb{R}$ such that $x_i(t) + R_0 \leq x \leq x_{i+1}(t) - R_0$, we have

$$|u(t,x)| \leq \sum_{i=1}^{N} Q_{c_i}(x - x_i(t)) + C \left\| u(t) - \sum_{i=1}^{N} Q_{c_i}(\cdot - x_i(t)) \right\|_{H^1} \leq C \sum_{i=1}^{N} e^{-\sqrt{c_i}R_0} + \eta_0.$$

Thus, for R_0 sufficiently large and for $\eta_0 > 0$ small enough being fixed, we have for some $T_1 > T_0$: for all $t \geq T_1$, for all $x \in [x_i(t) + R_0, x_{i+1}(t) - R_0]$,

$$\frac{2p}{p+1}|u(t,x)|^{p-1} \leq \frac{\nu}{4}.$$

If $x > x_{i+1}(t) - R_0$ or $x < x_i(t) + R_0$, then

$$\left| x - \frac{x_i(t) + x_{i+1}(t)}{2} \right| > \frac{x_{i+1}(t) - x_i(t)}{2} - R_0 > \frac{\nu t}{2} - R_0.$$

Consequently, for $t \geq T_1$ and $x \notin [x_i(t) + R_0, x_{i+1}(t) - R_0]$, we obtain

$$\left|\psi_i'(t,x)\right| \leq \frac{\sqrt{\nu}}{\pi} e^{-\frac{\sqrt{\nu}}{2}\left(\frac{x_{i+1}(t)-x_i(t)}{2} - R_0\right)} \leq C e^{-\frac{\nu^{\frac{3}{2}}}{4}t}.$$

We deduce that

$$\begin{aligned} \frac{d\mathcal{M}_i}{dt}(t) &\geq -\int_{\mathbb{R}} \left(3u_x^2 + \left(\delta_0 - \frac{\nu}{4}\right)u^2\right)\psi_i' - \int_{\mathbb{R}} \frac{\nu}{4}u^2|\psi_i'| - Ce^{-\frac{\nu^{\frac{3}{2}}}{4}t} \\ &\geq -\int_{\mathbb{R}} \left(3u_x^2 + \frac{\nu}{2}u^2\right)\psi_i' - Ce^{-\frac{\nu^{\frac{3}{2}}}{4}t} \geq -Ce^{-\frac{\nu^{\frac{3}{2}}}{4}t}. \end{aligned} \tag{3.94}$$

Similarly, we compute

$$\begin{aligned} \frac{d\tilde{\mathcal{E}}_i}{dt} = &-\int_{\mathbb{R}} \left[\left(u_{xx}^2 + u^p\right)^2 + 2u_{xx}^2 + \frac{x_i' + x_{i+1}'}{2}u_x^2 - \frac{x_i' + x_{i+1}'}{p+1}u^{p+1}\right](\psi_i)_x \, dx \\ &+ \int_{\mathbb{R}} u_x^2 \psi_i^{(3)} \, dx + 2p\int_{\mathbb{R}} u_x^2 u^{p-1}(\psi_i)_x \, dx + \kappa \frac{d\mathcal{M}_i}{dt} \\ \geq &-\nu \int_{\mathbb{R}} u_x^2(\psi_i)_x + \frac{\nu}{4}\int_{\mathbb{R}} u_x^2(\psi_i)_x + 2p\int_{\mathbb{R}} u_x^2 u^{p-1}(\psi_i)_x \\ &- \kappa \int_{\mathbb{R}} 3u_x^2\psi_i'(x) - Ce^{-\frac{\nu^{\frac{3}{2}}}{4}t} + \frac{x_i' + x_{i+1}'}{2}\int_{\mathbb{R}} \left(\frac{2}{p+1}u^{p+1} - \kappa u^2\right)(\psi_i)_x. \end{aligned} \tag{3.95}$$

As before, we can increase T_1 and reduce η_0 to have

$$2p\left|\int_{\mathbb{R}} u_x^2 u^{p-1}(\psi_i)_x\right| \leq \frac{\nu}{4}\int_{\mathbb{R}} u_x^2\,|(\psi_i)_x| + Ce^{-\frac{\nu^{\frac{3}{2}}}{4}t}$$

and

$$\left|\frac{2}{p+1}\int_{\mathbb{R}} u^{p+1}(\psi_i)_x\right| \leq \frac{\kappa}{2}\int_{\mathbb{R}} u^2\,|(\psi_i)_x|.$$

Eventually, this leads to

$$\begin{aligned}\frac{d\tilde{\mathcal{E}}_i}{dt} &\geq -\frac{3}{4}\nu\int_{\mathbb{R}} u_x^2(\psi_i)_x - Ce^{-\frac{\nu^{\frac{3}{2}}}{4}t} + \frac{\kappa(x_i' + x_{i+1}')}{4}\int_{\mathbb{R}} u^2\,|(\psi_i)_x| \\ &\geq \frac{3}{4}\nu\int_{\mathbb{R}} u_x^2\,|(\psi_i)_x| + \frac{\kappa\delta_0}{2}\int_{\mathbb{R}} u^2\,|(\psi_i)_x| - Ce^{-\frac{\sigma^{\frac{3}{2}}}{4}t} \geq -Ce^{-\frac{\nu^{\frac{3}{2}}}{4}t}.\Box\end{aligned}$$

Step 3: A Weinstein type functional

Let the functional $\mathcal{H}$ be given by

$$\mathcal{H} := \sum_{i=1}^{N}\frac{1}{c_i^2}\int_{\mathbb{R}}\left\{\partial_x\epsilon^2 + c_i\epsilon^2 - p\tilde{R}_i^{p-1}\epsilon^2\right\}\phi_i, \tag{3.96}$$

define

$$\mathcal{F} := \sum_{i=1}^{N}\frac{1}{c_i^2}\left\{\int_{\mathbb{R}}\left(\frac{1}{2}u_x^2 - \frac{1}{p+1}u^{p+1}\right)\phi_i + \frac{c_i}{2}\int_{\mathbb{R}} u^2\phi_i\right\}, \tag{3.97}$$

and set $w(t) := u(t) - \sum_{i=1}^{N} R_i(t)$, where for all $i = 1, \ldots, N$,

$$R_i(t) := Q_{c_i}(\cdot - x_i(t)).$$

We gather next some properties satisfied by $\mathcal{H}$ and $\mathcal{F}$ which are essential to obtain $\|\epsilon(t)\|_{H^1} = \mathrm{O}\left(e^{-\gamma t}\right)$ as $t \to +\infty$, for some $\gamma > 0$.

Lemma 3.7. *We have*

1. *(coercivity property satisfied by $\mathcal{H}$)*

$$\exists\,\lambda_0 > 0,\ \forall\, t \geq T, \quad \|\epsilon(t)\|_{H^1}^2 \leq \lambda_0\mathcal{H}(t) + \frac{1}{\lambda_0}\sum_{i=1}^{N}\left(\int_{\mathbb{R}}\epsilon(t)\tilde{R}_i(t)\right)^2; \tag{3.98}$$

2. *(expansion of $\mathcal{H}$)*

$$\mathcal{H} = 2\left(\mathcal{F} - \sum_{i=1}^{N}\frac{1}{c_i^2}\left\{\int_{\mathbb{R}}\left(\frac{1}{2}(\partial_x Q_{c_i})^2 - \frac{1}{p+1}Q_{c_i}^{p+1}\right) + \frac{c_i}{2}\int_{\mathbb{R}} Q_{c_i}^2\phi_i\right\}\right) + g, \tag{3.99}$$

where $|g(t)| \leq Ce^{-\frac{1}{4}\nu^{\frac{3}{2}}t} + C\tilde{\alpha}\|\epsilon(t)\|_{L^2}^2$;

3. *(second expression for $\mathcal{F}$)*

$$\mathcal{F} = \sum_{i=1}^{N-1}\left\{\left(\frac{1}{c_i^2}-\frac{1}{c_{i+1}^2}\right)\tilde{\mathcal{E}}_i + \left(\frac{1}{c_i}-\frac{1}{c_{i+1}}\right)\left(\frac{1}{2}-\kappa\left(\frac{1}{c_i}+\frac{1}{c_{i+1}}\right)\right)\mathcal{M}_i\right\} + \frac{1}{c_N^2}\tilde{\mathcal{E}}_N + \frac{1}{c_N}\left(\frac{1}{2}-\frac{\kappa}{c_N}\right)\mathcal{M}_N; \tag{3.100}$$

4. *(consequence of the monotonicity properties) For all $t' \geq t \geq T_1$, for all $i = 1, \ldots, N$,*

$$\mathcal{M}_i(t) - \int_{\mathbb{R}} Q_{c_i}^2 \leq 2\int_{\mathbb{R}} wR_i(t') + \int_{\mathbb{R}} w^2\phi_i(t') + Ce^{-\frac{\nu^{\frac{3}{2}}}{4}t}; \tag{3.101}$$

$$\tilde{\mathcal{E}}_i(t) - \int_{\mathbb{R}}\left\{\frac{1}{2}\left(\partial_x Q_{c_i}\right)^2 - \frac{Q_{c_i}^{p+1}}{p+1}\right\} \leq C\|w(t)\|_{L^\infty}\int_{\mathbb{R}} w^2\phi_i(t') - c_i\int_{\mathbb{R}} wR_i(t') + \frac{1}{2}\int_{\mathbb{R}}(w_x^2 - pR_i^{p-1}w^2)\phi_i(t') + Ce^{-\frac{\nu^{\frac{3}{2}}}{4}t}. \tag{3.102}$$

Proof of Lemma 3.7. We only give some indications, and particularly the key ingredients. Property (3.100) is obtained by Abel transformation. Now, we focus on the other lines.
Estimate (3.98) is a consequence of a localized version around ϕ_i of the coercivity property of the linearized operator around Q_{c_i} (for all $i = 1, \ldots, N$), which holds under the orthogonality conditions (3.80) satisfied by ϵ; we refer to [80, proof of Lemma 4].
To prove (3.99), one has obviously to replace ϵ by its definition (3.79).
To finish with, integrate the almost monotonicity properties as expressed in Lemma 3.6 between t and t' and use the expression of $u(t')$ in terms of $w(t')$ in order to obtain (3.101) and (3.102).
Let us mention furthermore that (3.98), (3.99), (3.101), and (3.102) rely all on classical inequalities used in studying quantities which are localized near the solitons, and which write in the present context as follows:

$$\forall\, i \neq j, \qquad (R_i(t,x) + |\partial_x R_i(t,x)|)\,\phi_j(t,x) \leq Ce^{-\frac{\nu^{\frac{3}{2}}}{4}t}e^{-\frac{\sqrt{\nu}}{4}|x-x_i(t)|} \tag{3.103}$$

$$\forall\, i, j, \qquad (R_i(t,x) + |\partial_x R_i(t,x)|)\,|\partial_x\phi_j(t,x)| \leq Ce^{-\frac{\nu^{\frac{3}{2}}}{4}t}e^{-\frac{\sqrt{\nu}}{4}|x-x_i(t)|} \tag{3.104}$$

$$\forall\, i, \qquad R_i(t,x)\,(1-\phi_i(t,x)) \leq Ce^{-\frac{\nu^{\frac{3}{2}}}{4}t}e^{-\frac{\sqrt{\nu}}{4}|x-x_i(t)|}. \square \tag{3.105}$$

Let us now explain how to conclude the proof. Note that (3.99), (3.100), (3.101), and (3.102) lead to: $\forall\, t' \geq t \geq T$,

$$
\begin{aligned}
\mathcal{H}(t) &\leq Ce^{-\frac{\nu^{\frac{3}{2}}}{4}t} + |g(t)| \\
&+ 2\sum_{i=1}^{N-1}\left(\frac{1}{c_i^2} - \frac{1}{c_{i+1}^2}\right)\left(C\|w(t')\|_{L^\infty}\int_{\mathbb{R}} w^2\phi_i(t') - c_i\int_{\mathbb{R}} w(t')R_i(t')\phi_i(t')\right. \\
&\qquad\left. + \frac{1}{2}\int_{\mathbb{R}}\left\{(\partial_x w)^2(t') - pR_i^{p-1}w^2(t')\right\}\phi_i(t')\right) \\
&+ 2\sum_{i=1}^{N-1}\left(\frac{1}{c_i} - \frac{1}{c_{i+1}}\right)\left(\frac{1}{2} - \kappa\left(\frac{1}{c_i} + \frac{1}{c_{i+1}}\right)\right)\left(2\int_{\mathbb{R}} wR_i(t') + \int_{\mathbb{R}} w^2\phi_i(t')\right) \\
&+ \frac{2}{c_N^2}\left(C\|w(t')\|_{L^\infty}\int_{\mathbb{R}} w^2\phi_N(t') + \frac{1}{2}\int_{\mathbb{R}}\left\{(\partial_x w)^2 - pR_N^{p-1}w^2(t')\right\}\phi_N(t')\right) \\
&+ \frac{2}{c_N}\left(\frac{1}{2} - \frac{\kappa}{c_N}\right)\left(2\int_{\mathbb{R}} wR_N(t') + \int_{\mathbb{R}} w^2\phi_N(t')\right) - \frac{2}{c_N}\int_{\mathbb{R}} wR_N(t'),
\end{aligned}
\tag{3.106}
$$

that is to: $\forall\, t' \geq t \geq T$,

$$
\begin{aligned}
\mathcal{H}(t) \leq \sum_{i=1}^{N}\frac{1}{c_i^2}&\left\{(\partial_x w)^2 + c_i w^2(t') - pR_i^{p-1}(t')w^2(t')\right\}\phi_i(t') \\
&+ C\|w(t')\|_{L^\infty}\|w(t')\|_{L^2}^2 + C\tilde{\alpha}\|\epsilon(t)\|_{L^2}^2 + Ce^{-\frac{\nu^{\frac{3}{2}}}{4}t},
\end{aligned}
\tag{3.107}
$$

where C does not depend on $\tilde{\alpha}$.

Remark 3.27. Note that the monotonicity property (3.101) can indeed be used in the preceding estimates since κ verifies $\kappa\left(\frac{1}{c_i} + \frac{1}{c_{i+1}}\right) < \frac{1}{2}$ and $\frac{\kappa}{c_N} < \frac{1}{2}$.

Assumption (3.59) tells us exactly that $\|w(t')\|_{H^1} \underset{t'\to+\infty}{\longrightarrow} 0$, thus we obtain

$$
\forall\, t \geq T, \quad \mathcal{H}(t) \leq Ce^{-\frac{\nu^{\frac{3}{2}}}{4}t} + C\alpha\|\epsilon(t)\|_{L^2}^2. \tag{3.108}
$$

Remark 3.28. Notice that it is important to consider w instead of ϵ in estimates (3.101) and (3.102) to obtain (3.108), thus to improve the a priori control of $\mathcal{H}$ by $\mathrm{O}\left(\|\epsilon\|_{H^1}^2\right)$.

Then, by (3.98) and by the following estimate

$$
\sum_{i=1}^{N}\left(\int_{\mathbb{R}}\epsilon(t)\tilde{R}_i(t)\right)^2 \leq Ce^{-\frac{\nu^{\frac{3}{2}}}{4}t}\|\epsilon(t)\|_{L^2} + C\tilde{\alpha}\|\epsilon(t)\|_{L^2}^2 \tag{3.109}
$$

(see Martel [63, Step 3]) for a proof), there exists $C_0 > 0$ such that for all $t \geq T$,

$$
\|\epsilon(t)\|_{H^1}^2 \leq Ce^{-\frac{\nu^{\frac{3}{2}}}{4}t} + C\tilde{\alpha}\|\epsilon(t)\|_{H^1}^2 + C\sum_{i=1}^{N}\left(\int_{\mathbb{R}}\epsilon(t)\tilde{R}_i(t)\right)^2 \leq C_0e^{-\frac{\nu^{\frac{3}{2}}}{4}t} + C_0\tilde{\alpha}\|\epsilon(t)\|_{H^1}^2.
$$

Now, due to the independence of C_0 with respect to $\tilde{\alpha}$, even if it means taking a smaller $\tilde{\alpha}$ so that $C_0\tilde{\alpha} < 1$, we infer

$$\forall\, t \geq T, \qquad \|\epsilon(t)\|_{H^1}^2 \leq Ce^{-\frac{\nu^{\frac{3}{2}}}{4}t}. \tag{3.110}$$

By (3.82), this implies that

$$\forall\, t \geq T, \qquad |x_i'(t) + y_i'(t) - c_i| \leq Ce^{-\frac{\nu^{\frac{3}{2}}}{8}t},$$

which leads to the existence of $y_i \in \mathbb{R}$ such that

$$x_i(t) + y_i(t) - c_i t \underset{t\to+\infty}{\longrightarrow} y_i$$

and

$$|x_i(t) + y_i(t) - c_i t - y_i| \leq Ce^{-\frac{\nu^{\frac{3}{2}}}{8}t}. \tag{3.111}$$

Hence, using (3.110), the triangular inequality, the following estimate

$$\|Q_{c_i}(\cdot - x_i(t) - y_i(t)) - Q_{c_i}(\cdot - c_i t - y_i)\|_{H^1} \leq C|x_i(t) + y_i(t) - c_i t - y_i|$$

(which is a consequence of Lemma 3.8 below), and (3.111), we have

$$\left\| u(t) - \sum_{i=1}^{N} Q_{c_i}(\cdot - c_i t - y_i) \right\|_{H^1} \leq Ce^{-\frac{\nu^{\frac{3}{2}}}{8}t}. \tag{3.112}$$

Lemma 3.8. *For all $i = 1, \ldots, N$, for all $r \geq 0$, and for all $s \in \mathbb{N}^*$,*

$$\left\|\partial_x^s Q_{c_i}(\cdot - r) - \partial_x^s Q_{c_i}\right\|_{L^2}^2 \leq \left(\|\partial_x^{s+1} Q_{c_i}\|_{L^2}^2 + (r + 2z_s)\|\partial_x^{s+1} Q_{c_i}\|_{L^\infty}^2\right) r^2,$$

where $z_s := \max\left(\left(\partial_x^{s+1} Q_{c_i}\right)^{-1}(\{0\}) \cap \mathbb{R}_+^*\right)$.

Proof of Lemmma 3.8. By the mean value theorem, we have:

$$\left\|\partial_x^s Q_{c_i}(\cdot - r) - \partial_x^s Q_{c_i}\right\|_{L^2}^2 = \int_{\mathbb{R}} \left(\partial_x^{s+1} Q_{c_i}(\xi_x)\right)^2 r^2\, dx,$$

where $x - r \leq \xi_x \leq x$ for all $x \in \mathbb{R}$.
Now, split the preceding integral into three regions: $x \leq -z_s$, $-z_s \leq x \leq z_s + r$, and $x \geq z_s + r$. In the first and third regions, use the monotonicity of $\left(\partial_x^{s+1} Q_{c_i}\right)^2$. We have:

$$\begin{cases} \left(\partial_x^{s+1} Q_{c_i}(\xi_x)\right)^2 \leq \left(\partial_x^{s+1} Q_{c_i}(x)\right)^2 & \text{if } x \leq -z_s \\ \left(\partial_x^{s+1} Q_{c_i}(\xi_x)\right)^2 \leq \left(\partial_x^{s+1} Q_{c_i}(x - r)\right)^2 & \text{if } x \geq z_s + r. \end{cases}$$

If $-z_s \leq x \leq z_s + r$, we have

$$\left(\partial_x^{s+1} Q_{c_i}(\xi_x)\right)^2 \leq \|\partial_x^{s+1} Q_{c_i}\|_{L^\infty}^2.$$

Thus, we obtain:

$$\left\|\partial_x^s Q_{c_i}(\cdot - r) - \partial_x^s Q_{c_i}\right\|_{L^2}^2 \leq \left(\|\partial_x^{s+1} Q_{c_i}\|_{L^2(|x|\geq z_s)}^2 + (r + 2z_s)\|\partial_x^{s+1} Q_{c_i}\|_{L^\infty}^2\right) r^2,$$

which puts an end to the proof. □

At the stage of (3.112), it suffices to see a posteriori that $\delta_0 \le c_1$ in order to obtain exactly (3.60). Let us justify it briefly.

For all $i \in \{1, \dots, N\}$, set $f_i(t,x) := Q_{c_i}(x - x_i(t)) - Q_{c_i}(x - c_i t - y_i)$.
From (3.59) and (3.112), we deduce

$$\left\| \sum_{i=1}^{N} f_i(t) \right\|_{H^1} \longrightarrow 0, \qquad \text{as } t \longrightarrow +\infty. \tag{3.113}$$

Define $th_p(x) := \tanh\left(\frac{p-1}{2}x\right)$. A direct computation yields

$$\forall\, x \in \mathbb{R}, \quad Q'(x) = -th_p(x) Q(x),$$

from which we have for all $i \in \{1, \dots, N\}$

$$\begin{aligned}\partial_x f_i(t,x) = &- \sqrt{c_i}\, th_p\big(\sqrt{c_i}(x - c_i t - y_i)\big) f_i(t,x) \\ &+ \sqrt{c_i}\Big(th_p\big(\sqrt{c_i}(x - x_i(t))\big) - th_p\big(\sqrt{c_i}(x - c_i t - y_i)\big)\Big) Q_{c_i}(x - x_i(t)).\end{aligned}$$

By (3.113), we have in particular

$$\int_{\mathbb{R}} \left(\sum_{i=1}^{N} \partial_x f_i(t,x) \right)^2 dx \longrightarrow 0, \qquad \text{as } t \longrightarrow +\infty.$$

Now, it follows from Lebesgue's dominated convergence theorem that

$$\left\| \sum_{i=1}^{N} \sqrt{c_i} f_i(t) \right\|_{L^2} \longrightarrow 0, \qquad \text{as } t \longrightarrow +\infty.$$

Combining this result with (3.113) and due to the fact that the speeds c_i are distinct two by two, we obtain successively for k describing the integers from $N-1$ to 1:

$$\left\| \sum_{i=1}^{k} \left(\sqrt{c_i} - \sqrt{c_{k+1}}\right) f_i(t) \right\|_{L^2} \longrightarrow 0, \qquad \text{as } t \longrightarrow +\infty.$$

Hence $\|f_1(t)\|_{L^2} \underset{t\to+\infty}{\longrightarrow} 0$ and even $\|f_1(t)\|_{H^1} \underset{t\to+\infty}{\longrightarrow} 0$ judging by the expression of $\partial_x f_1$. This implies $x_1(t) - c_1 t - y_1 \underset{t\to+\infty}{\longrightarrow} 0$. Now it is clear that condition $x_1'(t) \ge \delta_0$ forces to have $c_1 \ge \delta_0$.

3.5.2 Reformulation of the non dispersion property

Let us prove

Proposition 3.9. *Let $u \in \mathscr{C}([T_0, +\infty), L^2(\mathbb{R}))$. The following two assertions are equivalent:*

$$\exists\, \rho > 0, \qquad \sup_{t \ge T_0} \int_{x \le \rho t - R} u^2(t,x)\, dx \longrightarrow 0 \quad \text{as } R \longrightarrow +\infty. \tag{i}$$

and

$$\exists\, \sigma > 0, \qquad \int_{x \le \sigma t} u^2(t,x)\, dx \longrightarrow 0 \quad \text{as } t \longrightarrow +\infty. \tag{ii}$$

Proof. Let us first assume that assertion (i) holds. Fix $\sigma := \frac{\rho}{2}$ and let $\epsilon > 0$. By assumption, there exists $R > 0$ such that for all $t \geq T_0$,

$$\int_{x \leq \rho t - R} u^2(t,x)\ dx \leq \epsilon.$$

Then for all $t \geq \frac{R}{\sigma}$, we have

$$\int_{x \leq \sigma t} u^2(t,x)\ dx \leq \int_{x \leq \rho t - R} u^2(t,x)\ dx \leq \epsilon,$$

which exactly shows (ii).

Conversely, assuming (ii), we have for all $\epsilon > 0$ the existence of $T \geq T_0$ such that

$$\forall\ t \geq T, \quad \int_{x \leq \sigma t} u^2(t,x)\ dx \leq \epsilon.$$

Then, taking $\rho := \sigma$, for all $t \geq T$ and for all $R > 0$,

$$\int_{x \leq \rho t - R} u^2(t,x)\ dx \leq \epsilon.$$

It remains to observe that

$$\sup_{t \in [T_0,T]} \int_{x \leq \rho t - R} u^2(t,x)\ dx \longrightarrow 0 \quad \text{as } R \longrightarrow +\infty. \tag{3.114}$$

To this end, let us consider a non decreasing sequence of positive reals $(R_n)_{n \in \mathbb{N}}$ such that $R_n \longrightarrow +\infty$ and let us define on the compact set of times $[T_0, T]$

$$f_n : t \mapsto \int_{x \leq \rho t - R_n} u^2(t,x)\ dx.$$

We claim that for all $n \in \mathbb{N}$, f_n is continuous on $[T_0, T]$. Indeed, let us fix $t_0 \in [T_0, T]$ and write for $t \in [T_0, T]$:

$$f_n(t) - f_n(t_0) = \int_{x \leq \rho t - R_n} u^2(t,x)\ dx - \int_{x \leq \rho t - R_n} u^2(t_0,x)\ dx - \int_{\rho t - R_n}^{\rho t_0 - R_n} u^2(t_0,x)\ dx.$$

By the uniform boundedness theorem, we obtain that

$$\int_{\rho t - R_n}^{\rho t_0 - R_n} u^2(t_0,x)\ dx \longrightarrow 0 \quad \text{as } t \longrightarrow t_0.$$

In addition,

$$\left| \int_{x \leq \rho t - R_n} u^2(t,x)\ dx - \int_{x \leq \rho t - R_n} u^2(t_0,x)\ dx \right|$$
$$\leq \left(\int_{x \leq \rho t - R_n} |u(t,x) - u(t_0,x)|^2\ dx \right)^{\frac{1}{2}}$$

$$\times \left(\left(\int_{x \le \rho t - R_n} u^2(t,x)\ dx \right)^{\frac{1}{2}} + \left(\int_{x \le \rho t - R_n} u^2(t_0,x)\ dx \right)^{\frac{1}{2}} \right)$$
$$\le \|u(t) - u(t_0)\|_{L^2} \times 2 \sup_{s \in [T_0,T]} \|u(s)\|_{L^2}.$$

Since u belongs to $\mathscr{C}([T_0,+\infty), L^2(\mathbb{R}))$, the last term in the above estimate is finite and tends to 0 as $t \to t_0$. Thus,

$$f_n(t) - f_n(t_0) \to 0 \text{ as } t \to t_0$$

and f_n is continuous at t_0.

Moreover, the sequence of functions (f_n) converges pointwise to 0 on $t \in [T_0, T]$ and for all $t \in [T_0, T]$, $(f_n(t))_n$ is non-increasing. By application of a classical Dini lemma, we thus obtain the uniform convergence of (f_n) to 0 on $[T_0, T]$.

Hence, we have proved (3.114) and (i) holds true.

□

Chapter 4

Pointwise decay of the multi-solitons of the generalized Korteweg-de Vries equation

Abstract
Focusing on the decay properties of the multi-solitons of the generalized Korteweg-de Vries equations, we obtain that these solutions and their derivatives decrease exponentially in space on the left and inside the soliton region, by exploiting the general dynamics of the flow and asymptotic estimates available for the multi-solitons, respectively. On the right of the last solitary wave, we prove rapid decrease for all derivatives by a novel approach, based on an induction process and the good knowledge of the behavior of the multi-solitons in large time.

4.1 Introduction

4.1.1 Multi-solitons and main result

We are interested in *multi-solitons* of the generalized Korteweg-de Vries equations

$$\partial_t u + \partial_x(\partial_x^2 u + u^p) = 0 \tag{gKdV}$$

where (t, x) are elements of $\mathbb{R} \times \mathbb{R}$ and $p > 1$ is an integer.

Recall that (gKdV) admits a family of explicit traveling wave solutions indexed by $\mathbb{R}_+^* \times \mathbb{R}$. Let Q be the unique (up to translation) positive solution in $H^1(\mathbb{R})$ (known also as *ground state*) to the following stationary elliptic problem associated with (gKdV)

$$Q'' + Q^p = Q,$$

given by the explicit formula

$$Q(x) = \left(\frac{p+1}{2\cosh^2\left(\frac{p-1}{2}x\right)} \right)^{\frac{1}{p-1}}.$$

Then for all $c_0 > 0$ (velocity parameter) and $x_0 \in \mathbb{R}$ (translation parameter),

$$R_{c_0,x_0}(t,x) = Q_{c_0}(x - c_0 t - x_0) \tag{4.1}$$

is a global traveling wave solution of (gKdV) classically named *soliton* solution, where $Q_{c_0}(x) = c_0^{\frac{1}{p-1}} Q(\sqrt{c_0}x)$.

In this article, we explore some properties of the multi-solitons, built upon solitons and which are defined as follows:

Definition 4.1. *Let $N \geq 1$ and consider N solitons R_{c_i,x_i} as in* (4.1) *with speeds $0 < c_1 < \cdots < c_N$. A multi-soliton in $+\infty$ (resp. in $-\infty$) associated with the R_{c_i,x_i} is an H^1-solution u of (gKdV) defined in a neighborhood of $+\infty$ (resp. $-\infty$) and such that*

$$\left\| u(t) - \sum_{i=1}^{N} R_{c_i,x_i}(t) \right\|_{H^1} \to 0, \qquad \text{as } t \to +\infty \text{ (resp. as } t \to -\infty). \tag{4.2}$$

Recall that several results have been established so far concerning the multi-solitons of (gKdV).

For the original Korteweg-de Vries equation (with $p = 2$) and the modified Korteweg-de Vries equation (corresponding to $p = 3$), given N solitons $R_1, \ldots, R_N$ with velocity parameters $0 < c_1 < \cdots < c_N$ and translation parameters $x_1, \ldots, x_N \in \mathbb{R}$, the inverse scattering transform led to the existence of a solution u which is a multi-soliton both in $\pm\infty$ [89], with the same velocity parameters c_i in $\pm\infty$ but with distinct translation parameters; namely, there is a shift which one can quantify in terms of the c_i. Besides this solution u is explicit; for instance, for $p = 2$, it writes (see [32], [89, section 6], or [39]):

$$u = 6\frac{\partial^2}{\partial x^2} \ln \det M,$$

where $M(t,x)$ is the $N \times N$-matrix with generic entry

$$M_{(i,j)}(t,x) = \delta_{i,j} + 2\frac{(c_i c_j)^{\frac{1}{4}}}{\sqrt{c_i} + \sqrt{c_j}} e^{\frac{1}{2}(\sqrt{c_i}(x - c_i t) + x_i + \sqrt{c_j}(x - c_j t) + x_j)}$$

and $\delta_{i,j}$ is the Kronecker delta. We refer to Schuur [101, chapter 5, (5.5)] and to Lamb [55, chapter 5] for a formula when $p = 3$.

The study of multi-solitons is motivated by decomposition results obtained for general solutions of (gKdV) with $p = 2$ and $p = 3$; we refer to the soliton resolution properties stated in [8,28,101] for instance. Though multi-solitons have already been considered for various models, it is undoubtedly for the gKdV equations that the knowledge about multi-solitons has been most developed. One central result concerns the exhaustive classification of the multi-solitons according to the value of p.

Theorem 4.1 (Martel [63]; Côte, Martel and Merle [19]; Combet [11])**.** *Let $p > 1$ be an integer and let $N \geq 1$, $0 < c_1 < \cdots < c_N$, and $x_1, \ldots, x_N \in \mathbb{R}$.*
If $p \leq 5$, there exists $T_0 \geq 0$ and a unique multi-soliton $u \in \mathscr{C}([T_0, +\infty), H^1(\mathbb{R}))$ associated with the R_{c_i,x_i}, $i \in \{1, \ldots, N\}$.

If $p > 5$, there exists a one-to-one map Φ from $\mathbb{R}^N$ to the set of all H^1-solutions of (gKdV) defined in a neighborhood of $+\infty$ such that u is a multi-soliton in $+\infty$ associated with the R_{c_i,x_i} if and only if there exist $\lambda \in \mathbb{R}^N$ and $T_0 \geq 0$ such that $u_{|[T_0,+\infty)} = \Phi(\lambda)_{|[T_0,+\infty)}$.
Moreover, in each case, u belongs to $\mathscr{C}([T_0,+\infty), H^s(\mathbb{R}))$ for all $s \geq 0$, and there exist $\theta > 0$ (depending on the c_i) and positive constants λ_s such that for all $s \geq 0$, for all $t \geq T_0$,

$$\left\| u(t) - \sum_{i=1}^{N} R_{c_i,x_i}(t) \right\|_{H^s} \leq \lambda_s e^{-\theta t}. \tag{4.3}$$

In [30], we linked the multi-solitons of (gKdV) with the concept of *non dispersive* solutions which had already appeared in some form in early papers by Martel and Merle [65] in the context of the study of asymptotic stability of the solitons. For the Korteweg-de Vries equation in particular, we showed that non dispersion is a necessary and sufficient condition for an H^1-solution with sufficiently regular initial data to be a multi-soliton. More generally, we saw that an H^1-solution of (gKdV) which is non dispersive and remains close to a sum of N decoupled solitary waves for positive times is a multi-soliton in $+\infty$.

Non dispersion is actually significant insofar as it self improves to smoothness and exponential decay in space. That being said, it appears that non dispersion only allows for exponential decay of the solution on a half line $x < \rho t$ for some $\rho > 0$; to conclude to exponential decay everywhere, as for a sum of solitons, we had to assume in addition that the solution is global and that non dispersion holds also for negative times, which is a priori not verified by multi-solitons.

In this article, we go on studying qualitative properties of the multi-solitons by particularly concentrating on the behavior in space (at fixed time) of the multi-solitons of (gKdV).
Let us fix the parameters $0 < c_1 < \cdots < c_N$ and $x_1, \ldots, x_N \in \mathbb{R}$. We consider $T \in \mathbb{R}$ and $u \in \mathscr{C}([T,+\infty), H^1(\mathbb{R}))$ a multi-soliton of (gKdV) associated with the solitons R_{c_i,x_i}, $i = 1, \ldots, N$; we recall that u is unique if $p \leq 5$.

Let us state our main result.

Theorem 4.2. *Let $0 < \alpha < c_1$ and let $\beta > c_N$. Then there exist $T' > 0$, $\kappa_\alpha > 0$ and $\kappa_{\alpha,\beta} > 0$ such that for all $s \in \mathbb{N}$, there exists $C_s > 0$ such that for all $t \geq T'$,*
if $x \leq \alpha t$, (exponential decay at the left of the first soliton)

$$\left|\partial_x^s u(t,x)\right| \leq C_s e^{-\kappa_\alpha |x-\alpha t|}; \tag{4.4}$$

if $\alpha t \leq x \leq \beta t$, (exponential decay in the soliton region)

$$\left|\partial_x^s u(t,x)\right| \leq C_s \sum_{i=1}^{N} e^{-\kappa_{\alpha,\beta}|x-c_i t|}. \tag{4.5}$$

For all $s \in \mathbb{N}$ and for all $n \in \mathbb{N}$, there exists $C_{s,n} > 0$ such that for all $t \geq T$, for all $x > \beta t$, (algebraic decay at the right of the last soliton)

$$\left|\partial_x^s u(t,x)\right| \leq \frac{C_{s,n}}{(x-\beta t)^n}. \tag{4.6}$$

Remark 4.2. The constant κ_α can be chosen in $\left(0, \frac{\sqrt{\alpha}}{2}\right)$. The constant $\kappa_{\alpha,\beta}$ depends on α, β, and the velocity parameters c_i, $i = 1, \dots, N$; with θ as in (4.3), one can take

$$\kappa_{\alpha,\beta} = \min\left\{\sqrt{c_1}, \frac{\theta}{c_1 - \alpha}, \frac{\theta}{\beta - c_N}, \min_{i=1,\dots,N-1}\left\{\frac{\theta}{c_{i+1} - c_i}\right\}\right\}.$$

Theorem 4.2 shows in particular that the multi-solitons of (gKdV) belong to the Schwartz space $\mathscr{S}(\mathbb{R})$. To our knowledge, no similar result has yet emerged with regard to the multi-solitons of a nonintegrable partial differential equation. Since a sum of solitons and its derivatives decrease exponentially in space, obtaining exponential decay, at least at the left of the soliton region, is actually not surprising. As a corollary, we point out that multi-solitons are in particular non dispersive in the sense of [30]. Estimates (4.6) suggest that exponential decay holds also on the right, which would be in full conformity with the integrable setting, but this conclusion seems currently out of reach.

4.1.2 Comments

Exponential decay of the multi-solitons of (gKdV) at the left (on the region corresponding to $x \leq \alpha t$) follows from revisiting a strong monotonicity argument set up in [30, section 2] and originally developed by Laurent and Martel [57]. We underline that, rather than arising from non dispersion (which turns out to be a consequence of (4.4)), the produced monotonicity property holds for large values of t due to (4.2) and the decay of the solitons in the present context.

On an interval like $[\alpha t, \beta t]$, estimate (4.5) is established by means of the exponential decrease in time of the quantities $\left\|u(t) - \sum_{i=1}^N R_{c_i,x_i}(t)\right\|_{H^s}$, $s \in \mathbb{N}$, as stated in Theorem 4.1.

On the half line $x > \beta t$, it appears to be more difficult to show that multi-solitons decrease with exponential speed in space. The monotonicity argument, linked to the dynamic of the flow of (gKdV), does not apply anymore if the multi-soliton u is not assumed to be global. It is interesting to emphasize that multi-solitons (along with their derivatives) are proved to decrease faster than each polynomial function by exploiting the strong convergence result (4.3). This is achieved by a triangular process by considering the family of integrals

$$I_{s,n}(t) := \int_{x>\beta t} \left(\partial_x^s u\right)^2 (t,x)(x - \beta t)^n \, dx$$

for $s, n \in \mathbb{N}$. By induction on n, we succeed in establishing that, if for all $s \in \mathbb{N}$, $I_{s,n} < +\infty$, then for all $s \in \mathbb{N}$, $I_{s,n+1} < +\infty$. More precisely, assuming that $I_{0,n}, \dots, I_{s,n}, I_{s+1,n}$ are finite quantities, we prove that $I_{s,n+1}$ is also finite. From a technical point of view, this consists in one main new observation provided by this paper.

Actually this triangular way of obtaining algebraic decay for u and its derivatives comes from one term which carries a spatial derivative of u of order $s + 1$ and which appears in the derivative with respect to t of a functional of the form

$$\int_{\mathbb{R}} \left(\partial_x^s u\right)^2 \phi(t,x) \, dx,$$

where $\phi(t)$ denotes a certain weight function. This phenomenon is obviously related to the structure of (gKdV).

The algebraic decay then implies non dispersion of the multi-soliton on the right, which allows us to conclude to (**??**) by monotonicity if u is defined on $\mathbb{R}$.

Yet, we can expect the multi-solitons to decrease exponentially with respect to the space variable on the full line. This is still a very natural conjecture all the more so as multi-solitons are in some ways to be considered as special solutions in view of Theorem 4.1. Starting from (4.6), one possible first direction of research in order to prove this conjecture would be to "track" the constants λ_s that appear in (4.3). From another perspective, one could also rely on *Kato smoothing effect* [47] by using the gain of derivatives in space which arises from this concept.

Furthermore, by a monotonicity argument in the spirit of [30, section 2], we would show that it suffices to obtain exponential decay on the region $x \geq \beta t_0$ for some time t_0 in order to obtain exponential decay on the region $x \geq \beta t$ for all time $t \geq t_0$. This observation is in line with general statements linked with persistence of regularity and decay of solutions to (gKdV) previously developed by Kato [47] and Isaza, Linares and Ponce [40,41]. But similarly the question of obtaining exponential decrease everywhere in space for *one* time t_0 remains unresolved, even for the particular multi-soliton solutions; actually, it is unclear whether one can improve (4.6) to an analog of (4.4). Additionally, the decay on the left-hand side of the real line does not in general propagate forward in time; conversely, one can thus wonder if decay on the right-hand side does really propagate backward in time.

For the time being, estimate (4.6) is thus meaningful in the L^2-critical and supercritical cases. Besides, our argument could be adapted to other models such as the nonlinear Schrödinger equations (see the main theorems in [16]) in order to prove algebraic decay (in the sense of (4.6)) for the corresponding multi-solitons, especially when smoothness and asymptotic estimates hold and monotonicity properties are missing.

The next sections are devoted to the proof of Theorem 4.2.

4.2 Decay of the multi-solitons on the left

4.2.1 Decay of the multi-solitons on the left of the first soliton

Let $0 < \alpha < c_1$ and $\kappa_\alpha \in \left(0, \frac{\sqrt{\alpha}}{2}\right)$. The goal of this section is to prove

Proposition 4.3 (Exponential decay in large time on the left of the first soliton)**.** *There exists $T' \geq T$ such that for all $s \in \mathbb{N}$, there exists $C_s > 0$ such that for all $t \geq T'$, for all $x \leq \alpha t$,*

$$\left|\partial_x^s u(t,x)\right| \leq C_s e^{-\kappa_\alpha |x-\alpha t|}. \tag{4.7}$$

Proof. The proof follows the ideas of [30] and [57]. To reach the conclusion, we show the existence of $T' \in \mathbb{R}$ such that for each $k \in \mathbb{N}$, there exists $K_k > 0$ such that, with $\kappa := 2\kappa_\alpha$,

$$\forall\, t \geq T', \qquad \int_{x \leq \alpha t} \left(\partial_x^k u(t,x)\right)^2 e^{\kappa(\alpha t - x)}\, dx \leq K_k. \tag{4.8}$$

The first step is to obtain (4.8) for $k = 0$. For this, we claim a strong monotonicity property which is the purpose of Lemma 4.4 and Lemma 4.5 below.

Let us introduce, for some $\kappa > 0$ to be determined later, the function φ defined by

$$\varphi(x) = \frac{1}{2} - \frac{1}{\pi}\arctan(e^{\kappa x}).$$

It satisfies the following properties

$$\exists\, \lambda_0 > 0,\ \forall\, x \in \mathbb{R}, \qquad \lambda_0 e^{-\kappa|x|} < -\varphi'(x) < \frac{1}{\lambda_0} e^{-\kappa|x|}, \tag{4.9}$$

$$\forall\, x \in \mathbb{R}, \qquad |\varphi^{(3)}(x)| \leq -\kappa^2 \varphi'(x). \tag{4.10}$$

$$\exists\, \lambda_1 > 0,\ \forall\, x \geq 0, \qquad \lambda_1 e^{-\kappa x} \leq \varphi(x). \tag{4.11}$$

Moreover, let us observe that

$$\int_{x<\alpha t} u^2(t,x) e^{\kappa(\alpha t - x)}\, dx = \int_{x<0} u^2\,(t, x+\alpha t)\, e^{-\kappa x}\, dx, \tag{4.12}$$

and that, for all $x_0 < 0$,

$$\begin{aligned}\int_{x_0 \leq x < 0} u^2\,(t, x+\alpha t)\, e^{-\kappa x}\, dx &\leq e^{-\kappa x_0} \int_{x \geq x_0} u^2\,(t, x+\alpha t)\, e^{-\kappa(x-x_0)}\, dx \\ &\leq \frac{1}{\lambda_1} e^{-\kappa x_0} \int_{x \geq x_0} u^2\,(t, x+\alpha t)\, \varphi(x - x_0)\, dx \\ &\leq \frac{1}{\lambda_1} e^{-\kappa x_0} \int_{\mathbb{R}} u^2\,(t, x+\alpha t)\, \varphi(x - x_0)\, dx.\end{aligned} \tag{4.13}$$

Since $\kappa^2 < \alpha$, one can choose $\delta \in (0, \alpha - \kappa^2)$. We consider $T' \in \mathbb{R}$ to be determined later. Then, for fixed $t_0 \geq T'$ and $x_0 \in \mathbb{R}$, we define

$$\begin{array}{rccl} I_{(t_0,x_0)}: & [T', +\infty) & \to & \mathbb{R}^+ \\ & t & \mapsto & \displaystyle\int_{\mathbb{R}} u^2(t, x+\alpha t)\varphi\big(x - x_0 + \delta(t - t_0)\big)\, dx. \end{array}$$

We have

$$\forall\, t \geq T', \qquad I_{(t_0,x_0)}(t) = \int_{\mathbb{R}} u^2(t,x)\varphi\big(x - x_0 + \delta(t-t_0) - \alpha t\big)\, dx, \tag{4.14}$$

so that by derivation with respect to t, we obtain

$$\begin{aligned}\frac{dI_{(t_0,x_0)}}{dt}(t) = &-3\int_{\mathbb{R}} u_x^2(t,x)\varphi'(\tilde{x})\, dx - (\alpha - \delta)\int_{\mathbb{R}} u^2(t,x)\varphi'(\tilde{x})\, dx \\ &+ \int_{\mathbb{R}} u^2(t,x)\varphi^{(3)}(\tilde{x})\, dx + \frac{2p}{p+1}\int_{\mathbb{R}} u^{p+1}(t,x)\varphi'(\tilde{x})\, dx,\end{aligned} \tag{4.15}$$

where $\tilde{x} := x - x_0 + \delta(t - t_0) - \alpha t$.
We then claim

Lemma 4.4. *There exists $C_0 > 0$ such that for all $x_0 \in \mathbb{R}$, and for all $t_0, t \geq T'$,*

$$\frac{dI_{(t_0,x_0)}}{dt}(t) \geq -C_0 e^{-\kappa(-x_0 + \delta(t-t_0))}. \tag{4.16}$$

Proof. Due to property (4.10) of φ, we have

$$\left|\int_{\mathbb{R}} u^2(t,x)\varphi^{(3)}(\tilde{x})\ dx\right| \leq -\kappa^2 \int_{\mathbb{R}} u^2(t,x)\varphi'(\tilde{x})\ dx. \tag{4.17}$$

Furthermore we control the nonlinear part by considering

$$I_1(t) := \int_{|\tilde{x}|>-x_0+\delta(t-t_0)} u^{p+1}(t,x)\varphi'(\tilde{x})\ dx$$

and

$$I_2(t) := \int_{|\tilde{x}|\leq -x_0+\delta(t-t_0)} u^{p+1}(t,x)\varphi'(\tilde{x})\ dx.$$

On the one hand, we have due to (4.9)

$$\left|I_1(t)\right| \leq \frac{1}{\lambda_0} e^{-\kappa\left(-x_0+\delta(t-t_0)\right)} \int_{\mathbb{R}} |u|^{p+1}(t,x)\ dx \leq Ce^{-\kappa\left(-x_0+\delta(t-t_0)\right)}, \tag{4.18}$$

where we have used the Sobolev embedding $H^1(\mathbb{R}) \hookrightarrow L^{p+1}(\mathbb{R})$ and the fact that u belongs to $L^\infty([T',+\infty),H^1(\mathbb{R}))$. Note that $C>0$ is independent of x_0, t_0, and t.
On the other, we observe that

$$\begin{aligned}\left|I_2(t)\right| &\leq \|u(t)\|^{p-1}_{L^\infty\left(x\leq\alpha t\right)} \int_{x\leq\alpha t} u^2(t,x)|\varphi'(\tilde{x})|\ dx \\ &\leq \sqrt{2}^{p-1}\|u(t)\|^{\frac{p-1}{2}}_{L^2\left(x\leq\alpha t\right)}\|u_x(t)\|^{\frac{p-1}{2}}_{L^2\left(x\leq\alpha t\right)} \int_{\mathbb{R}} u^2(t,x)|\varphi'(\tilde{x})|\ dx \\ &\leq \sqrt{2}^{p-1}\|u(t)\|^{\frac{p-1}{2}}_{L^2\left(x\leq\alpha t\right)} \sup_{t\geq T'}\|u(t)\|^{\frac{p-1}{2}}_{H^1} \int_{\mathbb{R}} u^2(t,x)|\varphi'(\tilde{x})|\ dx.\end{aligned} \tag{4.19}$$

Since u is a multi-soliton, we can choose $T'\geq 0$ such that for all $t\geq T'$,

$$\sqrt{2}^{p-1}\|u(t)\|^{\frac{p-1}{2}}_{L^2\left(x\leq\alpha t\right)} \sup_{t'\geq T'}\|u(t')\|^{\frac{p-1}{2}}_{H^1} \leq \frac{p+1}{2p}(\alpha-\delta-\kappa^2). \tag{4.20}$$

Let us justify it briefly (it consists in the main change with respect to previous proofs based on non dispersion [30] or L^2-compactness [57]): we have

$$\begin{aligned}&\int_{x\leq\alpha t} u^2(t,x)\ dx \\ &\quad\leq 2\int_{x\leq\alpha t}\left(u-\sum_{i=1}^N R_{c_i,x_i}\right)^2(t,x)\ dx + 2\int_{x\leq\alpha t}\left(\sum_{i=1}^N R_{c_i,x_i}\right)^2(t,x)\ dx \\ &\quad\leq 2C_0^2 e^{-2\theta t} + 2N\sum_{i=1}^N \int_{x\leq\alpha t} R^2_{c_i,x_i}(t,x)\ dx\end{aligned}$$

and for all $i=1,\dots,N$, since $\alpha<c_i$, we have for $t\geq 0$:

$$\int_{x\leq\alpha t} R^2_{c_i,x_i}(t,x)\ dx \leq C\int_{x\leq\alpha t} e^{-\sqrt{c_i}|x-c_it-x_i|}e^{-\sqrt{c_i}|x-c_it-x_i|}\ dx$$

$$
\begin{aligned}
&\le C \int_{x\le \alpha t} e^{-\sqrt{c_i}(c_i-\alpha)t} e^{-\sqrt{c_i}|x-c_it-x_i|}\,dx\\
&\le Ce^{-\sqrt{c_i}(c_i-\alpha)t}\int_{\mathbb{R}} e^{-\sqrt{c_i}|x-c_it-x_i|}\,dx\\
&\le Ce^{-\sqrt{c_i}(c_i-\alpha)t}.
\end{aligned}
$$

where C denotes a positive constant which can change from one line to the other and which only depends on c_i (see expression (4.1)).
Thus, we can pick up $C \ge 0$ such that for all $t \ge 0$,

$$
\int_{x\le\alpha t} u^2(t,x)\,dx \le C\left(e^{-2\theta t} + \sum_{i=1}^{N} e^{-\sqrt{c_i}(c_i-\alpha)t}\right).
$$

Hence this leads to the existence of T' satisfying (4.20).

Taking into account (4.18), this eventually leads to the following estimate

$$
\frac{2p}{p+1}\left|\int_{\mathbb{R}} u^{p+1}(t,x)\varphi'(\tilde{x})\,dx\right| \le -(\alpha-\delta-\kappa^2)\int_{\mathbb{R}} u^2(t,x)\varphi'(\tilde{x})\,dx + C_0e^{-\kappa\left(-x_0-R+\delta(t-t_0)\right)}, \tag{4.21}
$$

where $C_0 := \frac{2p}{p+1}C$ is independent of x_0, t_0, and t. Gathering (4.17) and (4.21) in (4.15), we finally deduce

$$
\frac{dI_{(t_0,x_0)}}{dt}(t) \ge -3\int_{\mathbb{R}} u_x^2(t,x)\varphi'(\tilde{x})\,dx - C_0e^{-\kappa\left(-x_0-R+\delta(t-t_0)\right)}.
$$

Thus Lemma 4.4 is established. □

As a consequence of the preceding lemma,

$$
\exists\, C_1>0,\ \forall\, x_0\in\mathbb{R},\ \forall\, t\ge t_0, \qquad I_{(t_0,x_0)}(t_0) \le I_{(t_0,x_0)}(t) + C_1e^{\kappa x_0}, \tag{4.22}
$$

with C_1 independent of the parameters x_0 and t_0. Next, we claim the following:

Lemma 4.5. *For fixed $x_0\in\mathbb{R}$ and $t_0 \ge T'$, $I_{(t_0,x_0)}(t) \to 0$ as $t\to+\infty$.*

Proof. This lemma is shown by adapting the proof in [57, paragraph 2.1, Step 2] and in [30]. Let ε be a positive real number. As in the previous proof, because u is a multi-soliton, we can find $T_1 \ge T'$ large such that for all $t\ge T_1$,

$$
\int_{x<\alpha t} u^2(t,x)\,dx \le \frac{\varepsilon}{2}.
$$

Since $0\le\varphi\le 1$, this enables us to see that

$$
\int_{x<0} u^2\left(t,x+\alpha t\right)\varphi\big(x-x_0+\delta(t-t_0)\big)\,dx \le \int_{x<\alpha t} u^2(t,x)\,dx \le \frac{\varepsilon}{2}. \tag{4.23}
$$

Now, recall that φ is decreasing so that

$$
\int_{x\ge 0} u^2(t,x+\alpha t)\varphi\big(x-x_0+\delta(t-t_0)\big)\,dx
$$

$$\leq \varphi\big(-x_0+\delta(t-t_0)\big)\|u(t)\|_{L^2}^2$$
$$\leq \overline{C}\varphi\big(-\tilde{R}-x_0+\delta(t-t_0)\big), \tag{4.24}$$

with $\overline{C}=\|u(t)\|_{L^2}^2$ for all $t\in J$. Moreover, since $\varphi(x)\to 0$ as $x\to+\infty$, there exists $T_2\in\mathbb{R}$ such that for all $t\geq T_2$,

$$\overline{C}\varphi\big(-x_0+\delta(t-t_0)\big)\leq\frac{\varepsilon}{2}.$$

Then, for all $t\geq\max\{T_1,T_2\}$,

$$I_{(t_0,x_0)}(t)\leq\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.$$

Hence, we have finished proving Lemma 4.5. □

At this stage, we deduce from (4.22) and Lemma 4.5 that

$$\forall\, t_0\geq T',\ \forall\, x_0\in\mathbb{R},\quad I_{(t_0,x_0)}(t_0)\leq C_1e^{\kappa x_0}. \tag{4.25}$$

Thus, (4.13) leads to

$$\forall\, t\geq T',\quad \int_{x_0\leq x<0}u^2\,(t,x+\alpha t)\,e^{-\kappa x}\,dx\leq\frac{C_1}{\lambda_1}.$$

Letting $x_0\to-\infty$, we infer that

$$\forall\, t\geq T',\quad \int_{x<0}u^2\,(t,x+\alpha t)\,e^{-\kappa x}\,dx\leq\frac{C_1}{\lambda_1}.$$

which proves (4.8) with $k=0$.

Now, to conclude to (4.8) for all $k\in\mathbb{N}$, one actually proves by induction on $k\in\mathbb{N}$ the existence of $\tilde{K}_k\geq 0$ such that for all $t\geq T'$,

$$\int_{\mathbb{R}}\left(\partial_x^k u\right)^2(t,x+\alpha t)\,e^{-\kappa x}\,dx+\int_t^{t+1}\int_{\mathbb{R}}\left(\partial_x^k u\right)^2(s,x+\alpha s)\,e^{-\kappa x}\,dx\,ds\leq\tilde{K}_k. \tag{4.26}$$

For $k=0$, this is in fact a consequence of (4.8) and of the following estimate: for all $t\geq t_0\geq T'$,

$$I_{(t_0,x_0)}(t_0)-I_{(t_0,x_0)}(t)\leq\frac{C_1}{\lambda_1}e^{\kappa x_0}+3\int_{t_0}^t\int_{\mathbb{R}}u_x^2(s,x+\alpha s)\varphi'(x-x_0+\delta(s-t_0))\,dx\,ds$$

(which follows from the proof of Lemma 4.4).
Indeed, we notice that by (4.9) and since φ is decreasing, for $s\in[t_0,t]$,

$$\lambda_0e^{-\kappa|x-x_0|}<-\varphi'(x-x_0)\leq-\varphi'(x-x_0+\delta(s-t_0))$$

so that for $t=t_0+1$ in particular, we have

$$\int_{t_0}^{t_0+1}\int_{x_0<x}u_x^2(s,x+\alpha s)e^{-\kappa x}\,dx\,ds\leq Ce^{-\kappa x_0}\left(I_{(t_0,x_0)}(t)-I_{(t_0,x_0)}(t_0)\right)$$
$$\leq Ce^{-\kappa x_0}I_{(t_0,x_0)}(t_0+1)$$
$$\leq Ce^{-\kappa x_0}I_{(t_0+1,x_0)}(t_0+1)\leq C.$$

where the last inequality results from (4.25). Passing to the limit when $x_0\to-\infty$, we obtain the desired inequality (4.26).

The rest of the induction argument closely follows [57, paragraph 2.3 and paragraph 2.2 Step 2]. Since it does not depend on the properties of the multi-soliton and for the sake of brevity, we will not detail the proof (4.26) for higher values of k. □

4.2.2 Decay of the multi-solitons between the solitary waves

We consider $0 < \alpha < c_1$ and $\beta > c_N$. Let us prove

Proposition 4.6 (Exponential decay in the soliton region)**.** *There exists $\kappa_{\alpha,\beta} > 0$ such that for all $s \in \mathbb{N}$, there exists $C_s > 0$ such that for all $t \geq T$, for all $\alpha t \leq x \leq \beta t$,*

$$\left|\partial_x^s u(t,x)\right| \leq C_s \sum_{i=1}^{N} e^{-\kappa_{\alpha,\beta}|x-c_i t|}. \tag{4.27}$$

Proof. We denote $z(t) := u(t) - \sum_{i=1}^{N} R_{c_i,x_i}(t)$. For all $s \in \mathbb{N}$, for all $t \geq T$, we have by (4.3) and the Sobolev embedding $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$

$$\|\partial_x^s z(t)\|_{L^\infty} \leq C\|\partial_x^s z(t)\|_{H^1} \leq C\|z(t)\|_{H^{s+1}} \leq \lambda_{s+1} e^{-\theta t}.$$

Now, let $\kappa_{\alpha,\beta} := \min\left\{\sqrt{c_1}, \frac{\theta}{c_1-\alpha}, \frac{\theta}{\beta-c_N}, \min_{i=1,\dots,N-1}\left\{\frac{\theta}{c_{i+1}-c_i}\right\}\right\}$.
Fix $t \geq T$. For all $i = 1, \dots, N-1$ and $c_i t \leq x \leq c_{i+1} t$,

$$e^{-\theta t} \leq e^{-\kappa_{\alpha,\beta}(x-c_i t)} \quad \text{if and only if} \quad x \leq \left(c_i + \frac{\theta}{\kappa_{\alpha,\beta}}\right)t,$$

which is indeed satisfied since $c_{i+1} \leq c_i + \frac{\theta}{\kappa_{\alpha,\beta}}$ by the choice of $\kappa_{\alpha,\beta}$.
Similarly, for all $\alpha t \leq x \leq c_1 t$, we have $e^{-\theta t} \leq e^{-\kappa_{\alpha,\beta}(c_1 t - x)}$ because $c_1 \leq \alpha + \frac{\theta}{\kappa_{\alpha,\beta}}$.
And for all $c_N t \leq x \leq \beta t$, we have $e^{-\theta t} \leq e^{-\kappa_{\alpha,\beta}(x-c_N t)}$ because $\beta \leq c_N + \frac{\theta}{\kappa_{\alpha,\beta}}$.

Thus, we obtain that for all $t \geq T$ and for all $\alpha t \leq x \leq \beta t$,

$$\left|\partial_x^s z(t,x)\right| \leq \|\partial_x^s z(t)\|_{L^\infty} \leq \lambda_{s+1} e^{-\theta t} \leq \lambda_{s+1} \sum_{i=1}^{N} e^{-\kappa_{\alpha,\beta}|x-c_i t|}.$$

Moreover, for all $i = 1, \dots, N$, $\left|\partial_x^s R_{c_i,x_i}(t,x)\right| \leq C_{i,s} e^{-\sqrt{c_i}|x-c_i t|}$ for some $C_{i,s} > 0$ depending on i and s. Hence, we conclude to (4.27) by the triangular inequality and by the fact that $\kappa_{\alpha,\beta} \leq \sqrt{c_1}$. □

4.3 Decay of the multi-solitons on the right of the last soliton

In this subsection, we prove rapid decrease of the multi-soliton on the right. Let $\beta > c_N$. We state

Proposition 4.7 (Polynomial decay in large time on the right of the last soliton)**.** *For all $s \in \mathbb{N}$ and for all $n \in \mathbb{N}$, there exists $C_{s,n} > 0$ such that for all $t \geq T$, for all $x > \beta t$,*

$$\left(\partial_x^s u(t,x)\right)^2 \leq \frac{C_{s,n}}{(x-\beta t)^n}. \tag{4.28}$$

Proof. Let us denote $z(t) := u(t) - \sum_{i=1}^{N} R_{c_i,x_i}(t)$ and $R_i = R_{c_i,x_i}$. By Theorem 4.1, there exists $\theta > 0$ such that for all $s \in \mathbb{N}$, there exists $\lambda_s > 0$ such that for all $t \geq T$,

$$\|z(t)\|_{H^s} \leq \lambda_s e^{-\theta t}. \tag{4.29}$$

Set $\eta \in \left(\frac{c_1^3}{c_N^2}, c_1\right)$ and introduce the function $\phi : \mathbb{R} \to \mathbb{R}$ defined by $\phi(x) := \frac{2}{\pi}\arctan\left(e^{\sqrt{\eta}x}\right)$. We can check that

$$\forall\, x \in \mathbb{R}, \qquad \left|\phi^{(3)}(x)\right| \le \eta\phi'(x); \tag{4.30}$$

$$\forall\, x \in \mathbb{R}, \qquad 0 \le \phi'(x) \le Ce^{\sqrt{\eta}x}; \tag{4.31}$$

$$\forall\, x \le 0, \qquad 0 \le \phi(x) \le \frac{2}{\pi}e^{\sqrt{\eta}x}. \tag{4.32}$$

For $x_0 > 0$ and $t \ge T$, we consider $\tilde{x} = \tilde{x}(t) := x - x_0 - \beta t$. Notice that the choice of η is made in order to obtain the following interaction estimate, which roughly expresses that the growth of $x \mapsto e^{\sqrt{\eta}x}$ is weaker than the decay of the solitons.

Claim 4.3. *For all $k \in \mathbb{N}$, there exists $C = C(k) > 0$ such that for all $i = 1, \dots, N$ and for all $t \ge T$,*

$$\int_{\mathbb{R}} \left|\partial_x^k R_i(t,x)\right| (\phi(\tilde{x}) + \phi'(\tilde{x}))\ dx \le Ce^{-\sqrt{\eta}x_0}. \tag{4.33}$$

Proof of Claim 4.3. By (4.31) and (4.32), it suffices to show that

$$\int_{\mathbb{R}} \left|\partial_x^k R_i(t,x)\right| e^{\sqrt{\eta}(x-\beta t)}\ dx \le C.$$

On the one hand, we have

$$\begin{aligned}\int_{x\le\beta t} e^{-\sqrt{c_i}|x-c_it|}e^{\sqrt{\eta}(x-\beta t)}\ dx &\le \int_{x\le\beta t} e^{-\sqrt{c_i}|x-c_it|}\ dx\\ &\le \int_{\mathbb{R}} e^{-\sqrt{c_i}|x|}\ dx \le \frac{2}{\sqrt{c_i}}.\end{aligned}$$

On the other, since $\beta > c_N$ and $\beta\sqrt{\eta} < c_i\sqrt{c_i}$,

$$\begin{aligned}\int_{x>\beta t} e^{-\sqrt{c_i}|x-c_it|}e^{\sqrt{\eta}(x-\beta t)}\ dx &\le e^{(c_i\sqrt{c_i}-\beta\sqrt{\eta})t}\int_{x>\beta t} e^{(\sqrt{\eta}-\sqrt{c_i})x}\ dx\\ &\le \int_{x>\beta t} e^{(\sqrt{\eta}-\sqrt{c_i})x}\ dx \le \frac{e^{(\sqrt{\eta}-\sqrt{c_i})\beta t}}{\sqrt{c_i}-\sqrt{\eta}}.\end{aligned}$$

Hence, noticing that we also have $\left|\partial_x^k R_i(t,x)\right| \le C(i,k)e^{-\sqrt{c_i}|x-c_it|}$, Claim 4.3 holds. □

For all $s \in \mathbb{N}$, let us define for some $x_0 > 0$ and for all $t \ge T$:

$$J_{s,x_0}(t) := \int_{\mathbb{R}} \left(\partial_x^s z\right)^2 (t,x)\phi(x - x_0 - \beta t)\ dx.$$

We first show the following recurrence formula which makes the link between the functions J_{s,x_0}, $s \in \mathbb{N}$.

Lemma 4.8. *For all $s \in \mathbb{N}$, there exists $C_s \ge 0$ (independent of x_0) such that*

$$\left|\frac{d}{dt}J_{s,x_0}(t)\right| \le C_s\int_{\mathbb{R}}\sum_{k=1}^{s+1}\left(\partial_x^k z\right)^2 (t,x)\phi'(\tilde{x})\ dx + C_se^{-\gamma t}\sum_{k=0}^{s-1}J_{k,x_0}(t) + C_se^{-\sqrt{\eta}x_0}e^{-\theta t}. \tag{4.34}$$

Proof. Let us observe that

$$\begin{aligned}\frac{d}{dt}J_{s,x_0}(t) =& -3\int_{\mathbb{R}}\left(\partial_x^{s+1}z\right)^2\phi'(\tilde{x})\ dx\\ &+\int_{\mathbb{R}}\left(\partial_x^s z\right)^2\phi^{(3)}(\tilde{x})\ dx-\beta\int_{\mathbb{R}}\left(\partial_x^s z\right)^2\phi'(\tilde{x})\ dx\\ &+2\int_{\mathbb{R}}\partial_x^s\left(\left(z+\sum_{i=1}^N R_i\right)^p-\sum_{i=1}^N R_i^p\right)\left(\partial_x^s z\phi\right)_x\ dx.\end{aligned}\tag{4.35}$$

By (4.30), we have

$$\left|\int_{\mathbb{R}}\left(\partial_x^s z\right)^2\phi^{(3)}(\tilde{x})\ dx-\beta\int_{\mathbb{R}}\left(\partial_x^s z\right)^2\phi'(\tilde{x})\ dx\right|\le 2\beta\int_{\mathbb{R}}\left(\partial_x^s z\right)^2\phi'(\tilde{x})\ dx.\tag{4.36}$$

Let us control the nonlinear term $\int_{\mathbb{R}}\partial_x^s\left(z^p\right)\left(\partial_x^s z\phi\right)_x\ dx$ which does not contain any soliton. For $s=0$, we observe that

$$\int_{\mathbb{R}}z^p\left(z\phi\right)'\ dx=\frac{p}{p+1}\int_{\mathbb{R}}z^{p+1}\phi'(\tilde{x})\ dx.$$

Thus

$$\left|\int_{\mathbb{R}}z^p\left(z\phi\right)'\ dx\right|\le\frac{p}{p+1}\|z(t)\|_{L^\infty}^{p-1}\int_{\mathbb{R}}z^2\phi'(\tilde{x})\ dx\le C\int_{\mathbb{R}}z^2\phi'(\tilde{x})\ dx.\tag{4.37}$$

For $s\ge 1$, we can write

$$\begin{aligned}\int_{\mathbb{R}}\partial_x^s\left(z^p\right)\left(\partial_x^s z\phi\right)_x\ dx=&\int_{\mathbb{R}}\partial_x^s\left(z^p\right)\partial_x^s z\phi'(\tilde{x})\ dx\\ &-\int_{\mathbb{R}}\partial_x^{s-1}\left(z^p\right)\left(\partial_x^{s+2}z\phi+\partial_x^{s+1}z\phi'\right)\ dx.\end{aligned}$$

We have

$$\partial_x^k\left(z^p\right)=\sum_{i_1+\dots+i_p=k}\binom{k}{i_1,\dots,i_p}\partial_x^{(i_1)}z\dots\partial_x^{(i_p)}z$$

so that

$$\begin{aligned}\left|\int_{\mathbb{R}}\partial_x^{s-1}\left(z^p\right)\partial_x^{s+2}z\phi\ dx\right|&\le C\|\partial_x^{s+2}z(t)\|_{L^\infty}\sum_{i_1+\dots+i_p=s-1}\int_{\mathbb{R}}\left|\partial_x^{(i_1)}z\right|\dots\left|\partial_x^{(i_p)}z\right|\phi(\tilde{x})\ dx\\ &\le C\|z(t)\|_{H^{s+3}}\int_{\mathbb{R}}\sum_{k=0}^{s-1}\left(\partial_x^k z\right)^2\phi(\tilde{x})\ dx\\ &\le Ce^{-\theta t}\sum_{k=0}^{s-1}I_{k,t_0,x_0}(t),\end{aligned}\tag{4.38}$$

where we have used (4.29).

Similarly we obtain

$$\left|\int_{\mathbb{R}}\partial_x^{s-1}\left(z^p\right)\partial_x^{s+1}z\phi'\ dx\right|\le Ce^{-\theta t}\int_{\mathbb{R}}\sum_{k=0}^{s-1}\left(\partial_x^k z\right)^2\phi'(\tilde{x})\ dx\tag{4.39}$$

and

$$\left|\int_{\mathbb{R}} \partial_x^s (z^p)\, \partial_x^s z \phi' \, dx\right| \leq Ce^{-\theta t} \int_{\mathbb{R}} \sum_{k=0}^{s} \left(\partial_x^k z\right)^2 \phi'(\tilde{x}) \, dx. \tag{4.40}$$

Moreover,

$$\begin{aligned}
\int_{\mathbb{R}} \partial_x^s \left(\left(z + \sum_{i=1}^{N} R_i\right)^p - \sum_{i=1}^{N} R_i^p - z^p\right) \left(\partial_x^s z \phi\right)_x \, dx \\
\leq C\|z(t)\|_{H^{s+1}} \sum_{i=1}^{N} \sum_{k=0}^{s} \int_{\mathbb{R}} \left|\partial_x^k R_i(t,x)\right| \left(\phi(\tilde{x}) + \phi'(\tilde{x})\right) \, dx \\
\leq C\|z(t)\|_{H^{s+1}} e^{-\sqrt{\eta} x_0},
\end{aligned} \tag{4.41}$$

where the last line follows from Claim 4.3.
We obtain Lemma 4.8 by gathering the above estimates. □

Then, we obtain the following control of $J_{s,x_0}(t)$:

Lemma 4.9. *For all $s \in \mathbb{N}$, there exists $K_s \geq 0$ such that for all $t \geq T$:*

$$J_{s,x_0}(t) \leq K_s \int_t^{+\infty} \int_{\mathbb{R}} \left(\sum_{k=1}^{s+1} \left(\partial_x^k z\right)^2 (t',x) \phi'(\tilde{x}(t'))\right) \, dx \, dt' + K_s e^{-\sqrt{\eta} x_0} e^{-\theta t}. \tag{4.42}$$

Proof. The preceding lemma follows from (4.34) and an induction argument. Notice that for all $s \in \mathbb{N}$, $J_{s,x_0}(t)$ tends to 0 as t tends to $+\infty$. Thus, for $s = 0$, (4.42) follows by integration of (4.34) between t and $+\infty$. Now assume that (4.42) is proved for $0, \ldots, s-1$ for some particular $s \in \mathbb{N}^*$. Then, by integration of (4.34) between t and $+\infty$ (for $t \geq T$), it results:

$$\begin{aligned}
J_{s,x_0}(t) \leq{}& C_s \int_t^{+\infty} \int_{\mathbb{R}} \sum_{k=1}^{s+1} \left(\partial_x^k z\right)^2 (t',x) \phi'(\tilde{x}) \, dx \, dt' + C_s e^{-\sqrt{\eta} x_0} \int_t^{+\infty} e^{-\theta t'} dt' \\
&+ C_s \sum_{s'=0}^{s-1} K_{s'} \int_t^{+\infty} e^{-\theta t'} \int_{t'}^{+\infty} \int_{\mathbb{R}} \sum_{k=1}^{s'+1} \left(\partial_x^k z\right)^2 (t'',x) \phi'(\tilde{x}) \, dx \, dt'' \, dt' \\
&+ C_s \sum_{s'=0}^{s-1} K_{s'} e^{-\sqrt{\eta} x_0} \int_t^{+\infty} e^{-\theta t'} \, dt' \\
\leq{}& C_s \int_t^{+\infty} \int_{\mathbb{R}} \left(\sum_{k=1}^{s+1} \left(\partial_x^k z\right)^2 (t',x) \phi'(\tilde{x})\right) \, dx \, dt' \\
&+ \sum_{s'=0}^{s-1} C_s K_{s'} \left(\int_t^{+\infty} \int_{\mathbb{R}} \sum_{k=0}^{s'+1} \left(\partial_x^k z\right)^2 (t'',x) \phi'(\tilde{x}) \, dx \, dt''\right) \int_t^{+\infty} e^{-\theta t'} dt' \\
&+ \frac{\max\left\{C_s, \max\{C_s K_{s'}, s' = 0, \ldots, s-1\}\right\}}{\theta} e^{-\sqrt{\eta} x_0} e^{-\theta t},
\end{aligned}$$

hence the existence of $K_s \geq 0$ such that:

$$J_{s,x_0}(t) \leq K_s \int_t^{+\infty} \int_{\mathbb{R}} \left(\sum_{k=1}^{s+1} \left(\partial_x^k z\right)^2 (t',x) \phi'(\tilde{x})\right) \, dx \, dt' + K_s e^{-\sqrt{\eta} x_0} e^{-\theta t}.$$

□

Now, we show how to deduce from Lemma 4.9 the polynomial decay of z and its derivatives. This is the object of Claim 4.4 and Lemma 4.10 below.

Let us define $\Phi_{[-1]}(x) := \phi'(x)$ and for all $n \in \mathbb{N}$,

$$\Phi_{[n]}(x) := \int_{-\infty}^{x} \Phi_{[n-1]}(y)\, dy.$$

The following claim justifies that $\Phi_{[n]}$ is well-defined for all $n \in \mathbb{N}$ and in fact motivates the introduction of $\left(\Phi_{[n]}\right)_n$.

Claim 4.4. *We have for all $n \in \mathbb{N}$*

$$\begin{aligned} &\forall\, x \le 0, \qquad 0 \le \Phi_{[n]}(x) \le \frac{1}{\sqrt{\eta}^{n}} e^{\sqrt{\eta}x} \\ &\forall\, x \ge 0, \qquad \frac{1}{2}\frac{x^n}{n!} \le \Phi_{[n]}(x) \le \sum_{k=0}^{n} \frac{1}{\sqrt{\eta}^{n-k}} \frac{x^k}{k!} \end{aligned} \tag{4.43}$$

Proof. We argue by induction on n.
Note that $\Phi_{[0]} = \phi$ is an increasing function and that

$$\forall\, t \ge 0, \quad \arctan t \le t.$$

Thus

$$\begin{aligned} &\forall\, x \le 0, \qquad 0 \le \Phi_{[0]}(x) \le \frac{2}{\pi} e^{\sqrt{\eta}x} \le e^{\sqrt{\eta}x} \\ &\forall\, x \ge 0, \qquad \frac{1}{2} = \phi(0) \le \Phi_{[0]}(x) \le 1. \end{aligned}$$

Now assume that (4.43) holds for some $n \in \mathbb{N}$ being fixed. In particular, $\Phi_{[n]}$ is positive on $\mathbb{R}$ and is integrable in $-\infty$; thus $\Phi_{[n+1]}$ is well-defined. Moreover, by definition of $\Phi_{[n+1]}$ and by the induction assumption, we have for all $x \le 0$

$$0 \le \Phi_{[n+1]}(x) \le \int_{-\infty}^{x} \frac{1}{\sqrt{\eta}^{n}} e^{\sqrt{\eta}t}\, dt \le \frac{1}{\sqrt{\eta}^{n+1}} e^{\sqrt{\eta}x}.$$

In particular,

$$0 \le \Phi_{[n+1]}(0) \le \frac{1}{\sqrt{\eta}^{n+1}}.$$

By the induction assumption, we then infer that for all $x \ge 0$,

$$\Phi_{[n+1]}(x) = \Phi_{[n+1]}(0) + \int_0^x \Phi_{[n]}(t)\, dt$$

satisfies

$$0 + \int_0^x \frac{1}{2}\frac{t^n}{n!}\, dt \le \Phi_{[n+1]}(x) \le \frac{1}{\sqrt{\eta}^{n+1}} + \int_0^x \sum_{k=0}^{n} \frac{1}{\sqrt{\eta}^{n-k}} \frac{t^k}{k!}\, dt.$$

Thus for all $x \ge 0$,

$$\frac{1}{2}\frac{x^{n+1}}{(n+1)!} \le \Phi_{[n+1]}(x) \le \frac{1}{\sqrt{\eta}^{n+1}} + \sum_{k=0}^{n} \frac{1}{\sqrt{\eta}^{n+1-k}} \frac{x^{k+1}}{(k+1)!} = \sum_{k=0}^{n+1} \frac{1}{\sqrt{\eta}^{n+1-k}} \frac{x^k}{k!}.$$

This finishes the induction argument, hence the proof of Claim 4.4. □

Lemma 4.10. *For all $n \in \mathbb{N}$, for all $s \in \mathbb{N}$, there exists $K_{s,n} \geq 0$ such that for all $t \geq T$,*

$$\int_{\mathbb{R}} \left(\partial_x^s z\right)^2 \Phi_{[n]}(x - x_0 - \beta t)\ dx \leq K_{s,n} e^{-\theta t}.$$

Proof. The following inequality holds true: for all $s \in \mathbb{N}$,

$$\int_{\mathbb{R}} \left(\partial_x^s z\right)^2 \Phi_{[0]}(x - x_0 - \beta t)\ dx \leq K_{s,0} e^{-\theta t}. \tag{4.44}$$

Assume that for some $n \in \mathbb{N}$, we have:

$$\forall\, s \in \mathbb{N},\ \exists\, K_{s,n} \geq 0,\ \forall\, t \geq T,\ \forall\, x_n > 0,$$
$$\int_{\mathbb{R}} \left(\partial_x^s z(t,x)\right)^2 \Phi_{[n]}(x - x_n - \beta t)\ dx \leq K_{s,n} e^{-\theta t}. \tag{4.45}$$

Then take $s \in \mathbb{N}$ and $x_{n+1} > 0$. Integrate estimate (4.42) provided by Lemma 4.9 on $[x_1, +\infty)$ with respect to x_0, for some $x_1 > 0$. We obtain by Fubini theorem: for $t \geq T$,

$$\begin{aligned}
&\int_{\mathbb{R}} \left(\partial_x^s z\right)^2 (t,x) \int_{x_1}^{+\infty} \Phi_{[0]}(x - x_0 - \beta t)\ dx_0\ dx \\
&\quad \leq K_s \int_t^{+\infty} \int_{\mathbb{R}} \sum_{k=0}^{s+1} \left(\partial_x^k z\right)^2 (t',x) \int_{x_1}^{+\infty} \Phi_{[-1]}(x - x_0 - \beta t')\ dx_0\ dx\ dt' \\
&\quad\quad + \frac{K_s}{\sqrt{\eta}} e^{-\sqrt{\eta} x_1} e^{-\theta t}
\end{aligned}$$

and then by an affine change of variable

$$\begin{aligned}
&\int_{\mathbb{R}} \left(\partial_x^s z\right)^2 (t,x) \Phi_{[1]}(x - x_1 - \beta t)\ dx \\
&\quad \leq K_s \int_t^{+\infty} \int_{\mathbb{R}} \sum_{k=0}^{s+1} \left(\partial_x^k z\right)^2 (t',x) \Phi_{[0]}(x - x_1 - \beta t')\ dx\ dt' \\
&\quad\quad + \frac{K_s}{\sqrt{\eta}} e^{-\sqrt{\eta} x_1} e^{-\theta t}.
\end{aligned}$$

Then integrating the preceding estimate on $[x_2, +\infty)$ with respect to x_1, for some $x_2 > 0$ leads to

$$\begin{aligned}
&\int_{\mathbb{R}} \left(\partial_x^s z\right)^2 (t,x) \Phi_{[2]}(x - x_2 - \beta t)\ dx \\
&\quad \leq K_s \int_t^{+\infty} \int_{\mathbb{R}} \sum_{k=0}^{s+1} \left(\partial_x^k z\right)^2 (t',x) \Phi_{[1]}(x - x_2 - \beta t')\ dx\ dt' \\
&\quad\quad + \frac{K_s}{\sqrt{\eta}^2} e^{-\sqrt{\eta} x_2} e^{-\theta t}.
\end{aligned}$$

Iterating several integrations ($n+1$ in total), we finally obtain:

$$\int_{\mathbb{R}} \left(\partial_x^s z\right)^2 (t,x) \Phi_{[n+1]}(x - x_{n+1} - \beta t)\ dx$$

$$
\begin{aligned}
&\leq K_s \int_t^{+\infty} \int_{\mathbb{R}} \sum_{k=0}^{s+1} \left(\partial_x^k z\right)^2 (t',x)\Phi_{[n]}(x - x_{n+1} - \beta t')\ dx\ dt' \\
&\quad + \frac{K_s}{\sqrt{\eta}^{n+1}} e^{-\sqrt{\eta}x_{n+1}} e^{-\theta t} \\
&\leq K_s \sum_{k=0}^{s+1} K_{k,n} \int_t^{+\infty} e^{-\theta t'}\ dt' + \frac{K_s}{\sqrt{\eta}^{n+1}} e^{-\sqrt{\eta}x_{n+1}} e^{-\theta t} \\
&\leq \frac{K_s}{\theta} \sum_{k=0}^{s+1} K_{k,n} e^{-\theta t} + \frac{K_s}{\sqrt{\eta}^{n+1}} e^{-\sqrt{\eta}x_{n+1}} e^{-\theta t},
\end{aligned}
$$

where the second line follows from the induction assumption (4.45). Thus, there exists $K_{s,n+1} \geq 0$ such that for all $t \geq T$, for all $x_{n+1} > 0$,

$$
\int_{\mathbb{R}} \left(\partial_x^s z(t,x)\right)^2 \Phi_{[n+1]}(x - x_{n+1} - \beta t)\ dx \leq K_{s,n+1} e^{-\theta t}.
$$

This finishes proving Lemma 4.10.

□

Proposition 4.7 follows now from Lemma 4.10, from the Sobolev embedding $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$, and the decay of all derivatives of the R_i. □

Chapter 5

Asymptotic N-soliton-like solutions of the nonlinear Klein-Gordon equation

Abstract
We are interested in solutions of the nonlinear Klein-Gordon equation (NLKG) in $\mathbb{R}^{1+d}$, $d \geq 1$, which behave as a soliton or a sum of solitons in large time. In the spirit of other articles focusing on the supercritical generalized Korteweg-de Vries equations and on the nonlinear Schrödinger equations, we obtain an N-parameter family of solutions of (NLKG) which converges exponentially fast to a sum of N given (unstable) solitons. For $N = 1$, this family completely describes the set of solutions converging to the soliton considered; for $N \geq 2$, we prove uniqueness in a class with explicit algebraic rate of convergence.

5.1 Introduction

5.1.1 Setting of the problem

We consider the following nonlinear Klein-Gordon equation

$$\partial_t^2 u = \Delta u - u + f(u), \qquad \text{(NLKG)}$$

where u is a real-valued function of $(t, x) \in \mathbb{R} \times \mathbb{R}^d$ and f is a $\mathscr{C}^1$ real-valued function on $\mathbb{R}$. Let us denote by F the unique primitive of f on $\mathbb{R}$ which vanishes in 0. We make the following assumptions:

- if $d = 1$,
  (H1) f is odd and $f'(0) = 0$.
  (H2) There exists $r > 0$ such that $F(r) > \frac{1}{2}r^2$.
- if $d \geq 2$,
  (H'1) f is a pure H^1-subcritical nonlinearity $r \mapsto \lambda|r|^{p-1}r$, with $\lambda > 0$ and $p > 1$ if $d = 2$ and $p \in \left(1, \frac{d+2}{d-2}\right)$ if $d \geq 3$.

Ce chapitre fait l'objet d'un article soumis pour publication [31].

The (NLKG) equation classically rewrites as the following first order system in time:

$$\partial_t U = \begin{pmatrix} 0 & Id \\ \Delta - Id & 0 \end{pmatrix} U + \begin{pmatrix} 0 \\ f(u) \end{pmatrix}, \tag{NLKG'}$$

where U is the two-vector $\begin{pmatrix} u \\ \partial_t u \end{pmatrix}$.

Assumption **(H1)** for $d = 1$ or assumption **(H'1)** for $d \geq 2$ on the nonlinearity f ensures that the Cauchy problem is locally well-posed in the energy space $H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$ [35,92]. It is even globally well-posed if one assumes further sufficient smallness on the initial condition.

Recall also that the following quantities are conserved for $H^1 \times L^2$-solutions $(u, \partial_t u)$ of (NLKG'):

- the energy $\frac{1}{2} \int_{\mathbb{R}^d} \left\{ (\partial_t u)^2 + |\nabla u|^2 + u^2 - 2F(u) \right\} (t, x)\, dx$
- the momentum $\int_{\mathbb{R}^d} \{\partial_t u \nabla u\} (t, x)\, dx$.

Moreover, the structure of the equation is left invariant under the action of $\mathbb{R} \times \mathbb{R}^d$ by (time and space) translation, and under the action of the Lorentz group $O(1, d)$ which consists of the linear automorphisms of $\mathbb{R}^{1+d}$ that preserve the quadratic form $(t, x_1, \ldots, x_d) \mapsto t^2 - \sum_{i=1}^d x_i^2$. In other words, precising this latter action, for all $\beta \in \mathbb{R}^d$ with $\mathbb{R}^d$-euclidean norm $|\beta| < 1$ and $\gamma := \frac{1}{\sqrt{1-|\beta|^2}}$, u is a solution of (NLKG) if and only if

$$(t, x) \mapsto u\left(\Lambda_\beta(t, x)\right)$$

is still a solution to (NLKG), where Λ_β is the linear transformation with matrix

$$\begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta^\top & I_d + \frac{\gamma-1}{|\beta|^2}\beta^\top\beta \end{pmatrix}$$

in the canonical basis of $\mathbb{R}^{1+d}$. We observe in particular that

$$\Lambda_\beta(t, x) = (\gamma(t - \beta x), \gamma(x - \beta t))$$

if $d = 1$. We refer to [20] for further details concerning the Lorentz transformations in all dimensions.

It is well-known that (NLKG) admits a family of solitons indexed by two parameters: the velocity parameter $\beta \in \mathbb{R}^d$ with $|\beta| < 1$ and the translation parameter $x_0 \in \mathbb{R}^d$. Let Q denote the unique (up to translation) positive H^1 solution of the following stationary elliptic problem, associated with (NLKG):

$$\Delta Q - Q + f(Q) = 0 \tag{5.1}$$

which we take as radial; for the record, existence of Q follows from a standard result of Berestycki and Lions [2] due to **(H2)** or **(H'1)** and uniqueness has been proved in Kwong [54] (in the case where $f(u) = |u|^{p-1}u$ is the particular power nonlinearity) and in Serrin and Tang [103]. We recall that Q and its partial derivatives up to order 3 decay exponentially. Then for all $\beta \in \mathbb{R}^d$ such that $|\beta| < 1$, for all $x_0 \in \mathbb{R}^d$, the boosted ground state

$$Q_{\beta,x_0} : (t, x) \mapsto Q\left(pr \circ \Lambda_\beta(t, x - x_0)\right),$$

where $\gamma := \frac{1}{\sqrt{1-|\beta|^2}}$ and pr is the canonical projection $\mathbb{R}^{1+d} \longrightarrow \mathbb{R}^d$ on the last d coordinates, is a solution of (NLKG) known as *soliton*. In the one-dimensional case, this soliton rewrites

$$Q_{\beta,x_0} : (t,x) \mapsto Q\left(x - \beta t - x_0\right).$$

Soliton theory concerning (NLKG) has extensively been studied in many articles. One major result is linked to the classification of the solutions with energy near that of the ground state. Dynamics of the solutions u of (NLKG) on the threshold energy $E(u) = E(Q)$ has been investigated in Duyckaerts and Merle [24]. More generally, classification of the solutions with energy less than a quantity slightly larger than the energy of the ground state has been done by Nakanishi and Schlag [93] and by Krieger, Nakanishi and Schlag [53].

Let us also mention that solitons of (NLKG) are known to be orbitally unstable in $H^1(\mathbb{R}^d)$ by a general property by Grillakis, Shatah and Strauss [37].

We further develop soliton analysis by exploring solutions which behave as a soliton or a sum of solitons as time goes to infinity.

For all $\beta \in \mathbb{R}^d$ such that $|\beta| < 1$ and $x_0 \in \mathbb{R}^d$, let us denote

$$R_{\beta,x_0}(t,x) := \begin{pmatrix} Q_{\beta,x_0}(t,x) \\ \partial_t Q_{\beta,x_0}(t,x) \end{pmatrix} = \begin{pmatrix} Q_{\beta,x_0}(t,x) \\ -\beta \cdot \nabla Q_{\beta,x_0}(t,x) \end{pmatrix}.$$

When $x_0 = 0$, we will write R_β instead of $R_{\beta,0}$ for the sake of simplification.

Drawing on the work by Grillakis, Shatah and Strauss [37, 38], Côte and Muñoz [20] have developed and proved spectral results adapted to the unstable dynamic around the (vector) soliton R_β. Essential properties which are needed in this paper, as well as the introduction of useful notations, are presented in the next subsection. Note that a similar spectral theory was firstly considered by Pego and Weinstein [96] in the context of the generalized Korteweg-de Vries equations.

Starting from this point of view, we are interested in solutions which converge to a soliton or a sum of solitons for large values of t; these solutions are classically known as *multi-solitons*.

Let us consider an integer $N \geq 1$ and $2N$ parameters

$$x_1, \dots, x_N \in \mathbb{R}^d \quad \text{and} \quad \beta_1, \dots, \beta_N \in \mathbb{R}^d$$

such that

$$\forall\, i = 1, \dots, N, \quad |\beta_i| < 1 \qquad \text{and} \qquad \forall\, i \neq j, \quad \beta_i \neq \beta_j.$$

We recall the following theorem by Côte and Muñoz which states the existence of at least *one* multi-soliton.

Theorem 5.1 ([20]). *There exist σ_0, $t_0 \in \mathbb{R}$ and $C_0 > 0$, only depending on the sets $(\beta_i)_i$, $(x_i)_i$, and a solution $U = \begin{pmatrix} u \\ \partial_t u \end{pmatrix} \in \mathscr{C}([t_0, +\infty), H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d))$ of* (NLKG) *such that for all $t \geq t_0$,*

$$\left\| U(t) - \sum_{i=1}^N R_{\beta_i,x_i}(t) \right\|_{H^1 \times L^2} \leq C_0 e^{-\sigma_0 t}.$$

We notice that this theorem has been extended to solutions describing *multi-bound states* by Côte and Martel [18], that is to multi-traveling waves made of any number N of decoupled general (excited) bound states. In the present paper, we will however only focus on multi-solitary waves in the above sense.

Since solitons are unstable, a solution of (NLKG) which behaves as a soliton in large time is not expected to be necessarily a soliton. One of our goals is thus to precise the dynamic of the flow of (NLKG) near a soliton. Similarly, the dynamic near a sum of solitons is also supposed to be more complex as time goes to infinity.

5.1.2 Main results

Given N distinct velocity parameters, we aim at proving the existence of a whole family of multi-solitons which turns out to be the unique family of multi-solitons in a certain class of solutions. Our first result reads as follows.

Theorem 5.2. *Assume that f is of class $\mathscr{C}^2$ and $0 < |\beta_N| < \cdots < |\beta_1| < 1$. There exist $\sigma > 0$, $0 < e_{\beta_1} < \cdots < e_{\beta_N}$, $Y_{+,i} \in \mathscr{C}(\mathbb{R}, H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)) \cap L^\infty(\mathbb{R}, H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d))$ for $i = 1, \dots, N$ and an N-parameter family $(\varphi_{A_1,\dots,A_N})_{(A_1,\dots,A_N)\in\mathbb{R}^N}$ of solutions of* (NLKG) *such that, for all $(A_1, \dots, A_N) \in \mathbb{R}^N$, there exist $t_0 \in \mathbb{R}$ and $C > 0$ such that*

$$\forall\, t \geq t_0, \qquad \left\| \Phi_{A_1,\dots,A_N}(t) - \sum_{i=1}^N R_{\beta_i,x_i}(t) - \sum_{i=1}^N A_i e^{-e_{\beta_i} t} Y_{+,i}(t) \right\|_{H^1\times L^2} \leq C e^{-(\sigma + e_{\beta_N})t}, \quad (5.2)$$

where $\Phi_{A_1,\dots,A_N} := \begin{pmatrix} \varphi_{A_1,\dots,A_N} \\ \partial_t \varphi_{A_1,\dots,A_N} \end{pmatrix}$. In addition, if $(A'_1, \dots, A'_N) \neq (A_1, \dots, A_N)$, then $\varphi_{A'_1,\dots,A'_N} \neq \varphi_{A_1,\dots,A_N}$.

Remark 5.3. The parameters e_{β_i} and the functions $Y_{+,i}$ ($i = 1, \dots, N$) are defined in Proposition 5.9 and in subsection 5.1.5.
One can moreover precise the value of σ in Theorem 5.2; for this, we refer to (5.6).

Below is about classification in a class with polynomial decay. We emphasize that the corresponding polynomial decay rate is explicit and independent of the soliton parameters, which is undeniably a breakthrough with regard to the thorny question of the classification of multi-soliton solutions. As a comparison, we proved uniqueness of a multi-soliton in a class of solutions with decrease faster than any power of $\frac{1}{t}$ in the context of the nonlinear Schrödinger equations [16].

Theorem 5.4. *Under the assumptions of Theorem 5.2 and keeping the same notations, if u is a solution of* (NLKG) *such that*

$$\left\| U(t) - \sum_{i=1}^N R_{\beta_i,x_i}(t) \right\|_{H^1\times L^2} = \mathrm{O}\left(\frac{1}{t^\alpha}\right) \quad \text{as } t \to +\infty, \qquad (5.3)$$

where $U = \begin{pmatrix} u \\ \partial_t u \end{pmatrix}$ and where $\alpha > 3$, then there exist $A_1, \dots, A_N \in \mathbb{R}$ and $t_0 \in \mathbb{R}$ such that for all $t \geq t_0$, $U(t) = \Phi_{A_1,\dots,A_N}(t)$.

Remark 5.5. Notice that Theorem 5.2 and Theorem 5.4 apply in dimension $d \leq 5$ only. Indeed, assuming **(H'1)** and f of class $\mathscr{C}^2$ forces to have $p > 2$, hence $2 < \frac{d+2}{d-2}$ if $d \geq 3$.

In the case where only one soliton is considered, one can moreover improve the preceding Theorem by completely characterizing solutions which converge to a soliton in large time.

Theorem 5.6. *Let $\beta \in \mathbb{R}^d$, $|\beta| < 1$ and assume that f is of class $\mathscr{C}^2$.*
There exist $e_\beta > 0$, $Y_{+,\beta} \in \mathscr{C}(\mathbb{R}, H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d))$, and a one-parameter family $(u^A)_{A\in\mathbb{R}}$ of solutions of (NLKG) *such that for all $A \in \mathbb{R}$, there exists $t_0 = t_0(A) \in \mathbb{R}$ such that for all $t \geq t_0$*

$$\left\| U^A(t) - R_\beta(t) - Ae^{-e_\beta t} Y_{+,\beta}(t) \right\|_{H^1\times L^2} \leq Ce^{-2e_\beta t}, \tag{5.4}$$

where $U^A := \begin{pmatrix} u^A \\ \partial_t u^A \end{pmatrix}$. In addition, if $A \neq A'$, then $u^A \neq u^{A'}$.
Moreover, if u is a solution of (NLKG) *such that*

$$\| U(t) - R_\beta(t) \|_{H^1\times L^2} \to 0 \quad \text{as } t \to +\infty, \tag{5.5}$$

where $U = \begin{pmatrix} u \\ \partial_t u \end{pmatrix}$, then there exist $A \in \mathbb{R}$ and $t_0 \in \mathbb{R}$ such that for all $t \geq t_0$, $U(t) = U^A(t)$.

Remark 5.7. The parameter e_β (which depends on β) and the function $Y_{+,\beta}$ are defined in Proposition 5.9; they are intimately related to the spectral theory dealing with the flow around R_β.

It is interesting to remark that there are only three special solutions among the elements of the preceding family $(U^A)_{A\in\mathbb{R}}$, up to translations in time and in space. This is the object of the following

Corollary 5.1. *Consider the family of solutions $\left(U^A\right)_{A\in\mathbb{R}}$ defined in Theorem 5.6.*

1. *If $A > 0$, there exists $t_A \in \mathbb{R}$ such that for all possible t, $U^A(t) = U^1(t + t_A, \cdot + \beta t_A)$.*
2. *If $A < 0$, there exists $t_A \in \mathbb{R}$ such that for all possible t, $U^A(t) = U^{-1}(t + t_A, \cdot + \beta t_A)$.*
3. *For all $t \in \mathbb{R}$, $U^0(t) = R_\beta(t)$.*

Remark 5.8. Let us observe that Remark 5.5 is valid for Theorem 5.6 and Corollary 5.1 too.

Theorem 5.6 provides the behavior of the solutions converging to solitons at the order $\mathrm{O}\left(e^{-2e_\beta t}\right)$ in $H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$. The instability direction due to the existence of one negative eigenvalue (related to e_β) for the linearized operator around Q_β yields infinitely many possibilities (described by the real line) to perturb the soliton Q_β and to retrieve another solution of (NLKG) which however remains exponentially close in time (with decay rate e_β) to Q_β. Let us emphasize that this phenomenon appears quite naturally in the study of unstable solitons. In a stable mode such as for the L^2-subcritical generalized Korteweg-de Vries (gKdV) equation, a solution which converges to a soliton as time goes to infinity is pretty well-known to be exactly the corresponding soliton.

What is more, Theorem 5.6 is built exactly as in [10]. It reminds us of similar classification results, previously obtained for the L^2-supercritical gKdV equations by Combet [10, Theorem 1.1] and for the three-dimensional cubic Schrödinger equation by Duyckaerts and Roudenko [26, Proposition 3.1], both inspired from pioneering works of Duyckaerts and Merle [24, 25].

As far as the multi-soliton case is concerned, Theorem 5.2 provides an N-parameter family of solutions to (NLKG) which behave as a sum of N (differently boosted) solitons in large time. Here again, the existence result looks like that of Combet [11, 12] and reinforces the idea that, in general,

non-uniqueness holds for multi-solitons in an unstable context. Furthermore, Theorem 5.4 opens the way to treat the question of the classification of multi-solitons for other models, at least in the restraint class of solutions with algebraic convergence in the sense of (5.3).
Regarding the global approach developed in this article, we shed new light on the construction of the one-parameter family $(U^A)_{A\in\mathbb{R}}$ by a compactness procedure. Note that the special solution U^{-1} in [10] has also been obtained in a first rigorous way by a compactness method (the starting point was the proof of instability of the soliton), but the question of obtaining U^1 (as well as the other solutions U^A for $A > 0$) by such method was raised (see Remark 4.15 in [10]) and, to our knowledge, remained unanswered. Obviously, our process can be thought and used as an alternative to prove similar results in the context of other partial differential equations which involve unstable solitons, and for which the spectral theory around the ground states is well understood (and actually analogous to the present case). This is a nice feature of our paper.

A second interesting point to be discussed is about proving uniqueness of multi-solitary waves. Exploring the possibility of obtaining a "weak" monotonicity property like (5.42) in Corollary 5.11 would be a promising direction of research in order to classify multi-solitons of other models.

Besides, a significant issue would be to know to what extent the classification obtained in Theorem 5.6 could be transcribed to the multi-soliton topic. Indeed, contrary to the gKdV setting [10] and especially as for the NLS case [12, 16], proving general uniqueness in the sense of (5.5), where R_β is replaced by a sum of several solitons R_{β_i,x_i}, still remains unclear.

Another question to be addressed could be related to the potential generalizations of the previous theorems to multi-bound states.

5.1.3 Outline

In the spirit of [63] (dealing with the construction of multi-soliton solutions), our approach to constructing the families in Theorem 5.2 and Theorem 5.6 is based on backward uniform $H^1 \times L^2$-estimates satisfied by well-chosen sequences of solutions of (NLKG) which aim to approximate the desired solutions. We entirely exploit a coercivity property available in the present matter (see Proposition 5.10 stated below) in order to establish those estimates and, in fact, to obtain the expected exponential convergences to zero in time. We finally use continuity of the flow of (NLKG) for the weak $H^1 \times L^2$-topology to obtain special solutions which fulfill (5.2) or (5.4).

The proof of Theorem 5.2 is thus based on compactness and energy methods; it follows the strategy of [11, 12]. The construction of $\Phi_{A_1,\dots,A_N}$ is done by iteration, by means of Proposition 5.12 which roughly asserts that each multi-soliton can be perturbed slightly at the order $e^{-e_{\beta_j}t}$ around the soliton R_{β_j,x_j}. In order to establish this key proposition, we particularly rely on the topological ingredient set up originally by Côte, Martel and Merle [19] for the construction of one given multi-soliton in unstable situations.

For the construction of U^A in the one-soliton case (Theorem 5.6), we also substantially rely on the spectral theory available for (NLKG), built on the linearized operator of the flow around the boosted ground state. In particular, we point out that our approach differs from previous articles [10, 24–26] where the construction centers around the contraction principle.

Regarding the question of classification, from (5.4) and orthogonality properties exposed in the next subsection, we notice that A corresponds to the limit of $e^{e_\beta t}\langle U^A, Z_{-,\beta}\rangle$ as $t \to +\infty$ in Theorem 5.6. This is precisely useful for the uniqueness part of this theorem, where the goal is to identify

U with an element of the one-parameter family $(U^A)_{A\in\mathbb{R}}$. Actually, to prove the second part of Theorem 5.6, we follow [10] in the first instance (up to the obtainment of an exponential control of $U - R_\beta$). Then, a refined version of the coercivity argument considered in [10], and indeed an elementary but careful analysis of the available estimates, allows us to reach the conclusion, that is to show that U equals some U^A (already constructed in the first part of the theorem), and that without making use of any supplementary tool. We underline once again that we do not need any fixed point argument to conclude.

Yet, we do not exclude the possibility to prove Theorem 5.6 via the contraction principle; if the nonlinearity f is sufficiently regular ($\mathscr{C}^{s+1}$ for instance), one could precise by this means the behavior of U^A in large time, and indeed expand the solution at the order $O\left(e^{-(s+1)e_\beta t}\right)$ in the Sobolev space H^s, in line with [24]. For clarity purposes, we will not explore this path further and anyway, our description of the family $(U^A)_{A\in\mathbb{R}}$ is sufficient to characterize solutions verifying (5.5).

Concerning Theorem 5.4, the identification of the solution satisfying (5.3) is done step by step as in [11]. The core of the proof is the obtainment of an almost monotonicity property, inspired by Martel and Merle [76]. By means of a technical lemma of analysis (we refer to Lemma 5.26 in Appendix), this monotonicity property allows us to see that any multi-soliton in the class with polynomial convergence to zero converges in fact exponentially (see subsection 5.3.1), provided one assumes suitable integrability conditions in the neighborhood of $+\infty$ (and indeed $\alpha > 3$). Such a "weak" monotonicity property has a priori been used so far only for the construction of multi-solitons or multi-bound states of the energy-critical wave equation [76,110,111]. We also underline that, by Lemma 5.26, we directly obtain the adequate exponential convergence rate which allows us to identify A_1; this is in contrast with [11]. The monotonicity property is also used as a key ingredient to identify the other parameters $A_2, \ldots, A_N$.

This paper is organized as follows. In Section 2, we introduce essential notations and tools which are used throughout our article. In the following sections, we establish the proofs of our main results; for ease of writing, each proof will be made in dimension 1. Section 3 is devoted to the construction of the family of multi-solitons described in Theorem 5.2. Section 4 deals with the classification of multi-solitons, which is the object of Theorem 5.4. In Section 5, we study the existence part of Theorem 5.6 focusing on one soliton. Section 6 aims at proving the second part of this latter theorem, that is general uniqueness of the one-parameter family previously constructed. In the appendix, we explain how to adapt the proof to all dimensions and we justify Corollary 5.1, and we state and prove the lemma of analytic theory of differential equations used in Section 4.

As usual, C denotes a positive constant which may depend on the soliton parameters and change from one line to the other, but which is always independent of t and x.

5.1.4 Acknowledgments

The author would like to thank his supervisor Raphaël Côte for his constant encouragements and for fruitful discussions.

5.1.5 Notations, review of spectral theory, and multi-solitons

Elements of spectral theory concerning (NLKG)

For all $U = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, V = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$, we define the scalar product:

$$\langle U, V \rangle := \int_{\mathbb{R}^d} (u_1 v_1 + u_2 v_2)\, dx$$

and the energy norm

$$\|U\|_{H^1 \times L^2} := \left(\|u_1\|_{H^1}^2 + \|u_2\|_{L^2}^2 \right)^{\frac{1}{2}}.$$

Under assumption **(H'1)** (or **(H1)** and **(H2)** in the particular one-dimensional case), the operator $L := -\Delta + Id - f'(Q)$ admits a unique simple negative eigenvalue, which we denote by $-\lambda_0$. The kernel of L is spanned by $(\partial_{x_i} Q)_{i=1,\dots,d}$ [62, 84].
Note that for a general nonlinearity f and for $d \geq 2$, the operator L possibly counts several and multiple negative eigenvalues. We refer to Côte and Martel [18] for the detail of the spectral properties in this case.

With a slight abuse of notation, we still denote by Q_β the function defined on $\mathbb{R}^d$ by $Q_\beta(x) := Q(\gamma x)$, where $\gamma = \frac{1}{\sqrt{1-|\beta|^2}}$ so that for all t, $Q_\beta(x) = Q_\beta(t,x)$. In the sequel, we sometimes omit the variables x and t when there is no ambiguity (we work with functions which either depend on time or not).

For all $\beta \in \mathbb{R}^d$ with $|\beta| < 1$, we consider the matrix operator

$$H_\beta := \begin{pmatrix} -\Delta + Id - f'(Q_\beta) & -\beta \cdot \nabla \\ \beta \cdot \nabla & Id \end{pmatrix},$$

the matrix $J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and the operator

$$\mathscr{H}_\beta := -H_\beta J = \begin{pmatrix} -\beta \cdot \nabla & \Delta - Id + f'(Q_\beta) \\ Id & \beta \cdot \nabla \end{pmatrix}.$$

We define for all $j = 1, \dots, d$

$$Z_{j,\beta} := \begin{pmatrix} -\beta \cdot \nabla \left(\partial_{x_j} Q_\beta \right) \\ \partial_{x_j} Q_\beta \end{pmatrix}.$$

Proposition 5.9 (Côte and Muñoz [20])**.** *We have $\mathscr{H}_\beta Z_{j,\beta} = 0$ for all $j \in \{1, \dots, d\}$ and there exist two functions $Z_{\pm,\beta}$ whose components decrease exponentially in space and such that*

$$\mathscr{H}_\beta Z_{\pm,\beta} = \pm e_\beta Z_{\pm,\beta}$$

where $e_\beta := \frac{\sqrt{\lambda_0}}{\gamma}$.
Moreover there exist unique functions $Y_{\pm,\beta}$ (whose components are exponentially decreasing in space) such that

$$H_\beta Y_{\pm,\beta} \in \operatorname{Span}\{Z_{\pm,\beta}\}, \quad \langle J Z_{j,\beta}, Y_{\pm,\beta} \rangle = 0, \quad \textit{and} \quad \langle Y_{\pm,\beta}, Z_{\mp,\beta} \rangle = 1.$$

In addition, the following orthogonality properties hold:

$$\langle Y_{\pm,\beta}, Z_{\pm,\beta}\rangle = 0 \quad \text{and} \quad \langle JZ_{0,\beta}, Z_{\pm,\beta}\rangle = 0.$$

The following coercivity property turns out to be a crucial tool in our paper.

Proposition 5.10 (Almost coercivity of H_β; Côte and Muñoz [20])**.** *There exists $\mu > 0$ such that for all $V \in H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$,*

$$\langle H_\beta V, V\rangle \geq \mu \|V\|^2_{H^1\times L^2} - \frac{1}{\mu}\left[\langle V, Z_{+,\beta}\rangle^2 + \langle V, Z_{-,\beta}\rangle^2 + \sum_{j=1}^{d}\langle V, JZ_{j,\beta}\rangle^2\right].$$

Multi-soliton results

Let us consider a set of $2N$ parameters as given in Theorem 5.2 and the associated (vector) solitons $R_i = \begin{pmatrix} Q_i \\ \partial_t Q_i \end{pmatrix} := R_{\beta_i,x_i}$, $i = 1,\dots,N$. We introduce moreover the vectors:

- $Y_{\pm,i}(t,x) := Y_{\pm,\beta_i}\left(pr \circ \Lambda_{\beta_i}(t, x - x_i)\right)$
- $Z_{\pm,i}(t,x) := Z_{\pm,\beta_i}\left(pr \circ \Lambda_{\beta_i}(t, x - x_i)\right)$,

where $\gamma_i := \frac{1}{\sqrt{1-|\beta_i|^2}}$. We denote $e_i = e_{\beta_i} := \frac{\sqrt{\lambda_0}}{\gamma_i} = \sqrt{\lambda_0(1-|\beta_i|^2)}$.

In particular, let us observe that for all $i = 1,\dots,N$, $Y_{\pm,i}$ belongs to $\mathscr{C}(\mathbb{R}, H^1(\mathbb{R}^d)\times L^2(\mathbb{R}^d)) \cap L^\infty(\mathbb{R}, H^1(\mathbb{R}^d)\times L^2(\mathbb{R}^d))$.

There exists $\ell \in \mathbb{R}^d$ such that

$$\forall\, i \neq j, \quad \ell\cdot\beta_i \neq \ell\cdot\beta_j \qquad \text{and} \qquad \forall\, i, \quad |\ell\cdot\beta_i| < 1;$$

we postpone the argument in the appendix. (If $d = 1$, one can take obviously $\ell = 1$.) Let us consider the permutation η of $\{1,\dots,N\}$ such that

$$-1 < \ell\cdot\beta_{\eta(1)} < \cdots < \ell\cdot\beta_{\eta(N)} < 1.$$

We denote also

$$\sigma := \frac{1}{16}\min\left\{e_1, \gamma_N \min\{\ell\cdot(\beta_{\eta(2)} - \beta_{\eta(1)}), \dots, \ell\cdot(\beta_{\eta(N)} - \beta_{\eta(N-1)})\}\right\} > 0. \tag{5.6}$$

We can quantify the interactions between the solitons R_i and the functions $Y_{\pm,i}$ and $Z_{\pm,i}$, for $i = 1,\dots,N$ in terms of the parameter σ. This is the object of the following

Proposition 5.11. *We have for all $i \neq j$, for all $k, l \in \{0,1,2\}$, and for all $t \geq 0$,*

$$\left\langle \partial_x^k R_i(t), \partial_x^l R_j(t)\right\rangle = O\left(e^{-4\sigma t}\right).$$

$$\left\langle Y_{\pm,i}(t), Y_{\pm,j}(t)\right\rangle = O\left(e^{-4\sigma t}\right).$$

$$\left\langle Z_{\pm,i}(t), Z_{\pm,j}(t)\right\rangle = \mathrm{O}\left(e^{-4\sigma t}\right).$$

$$\left\langle Y_{\pm,i}(t), \partial_x^l R_j(t)\right\rangle = \mathrm{O}\left(e^{-4\sigma t}\right).$$

$$\left\langle Z_{\pm,i}(t), \partial_x^l R_j(t)\right\rangle = \mathrm{O}\left(e^{-4\sigma t}\right).$$

$$\left\langle Y_{\pm,i}(t), Z_{\pm,j}(t)\right\rangle = \mathrm{O}\left(e^{-4\sigma t}\right).$$

What is more, due to Theorem 5.1, there exist $t_0 \in \mathbb{R}$ and $C > 0$, only depending on the sets $(\beta_i)_i$, $(x_i)_i$, and a solution $\Phi_0 = \begin{pmatrix} \varphi_0 \\ \partial_t \varphi_0 \end{pmatrix} \in \mathscr{C}([t_0, +\infty), H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d))$ of (NLKG) such that for all $t \geq t_0$,

$$\left\| \Phi_0(t) - \sum_{i=1}^N R_i(t) \right\|_{H^1 \times L^2} \leq C e^{-4\sigma t}. \tag{5.7}$$

When dealing with the multi-soliton case, we will need to consider in the present article the euclidean space $\left(\mathbb{R}^k, |\cdot|\right)$ and euclidean balls and spheres of radius $r > 0$ in $\mathbb{R}^k$, $k = 1, \ldots, N$; in particular we define:

$$B_{\mathbb{R}^k}(r) := \{x \in \mathbb{R}^k \mid |x| \leq r\}$$
$$S_{\mathbb{R}^k}(r) := \{x \in \mathbb{R}^k \mid |x| = r\}.$$

5.2 Construction of a family of multi-solitons for $N \geq 2$

In this section, we give a detailed proof of Theorem 5.2 in the one-dimensional case.
Let $N \geq 2$ and $x_1, \ldots, x_N$, $\beta_1, \ldots, \beta_N$ be $2N$ parameters as in Theorem 5.2. Denote by φ a multi-soliton solution associated with these parameters, satisfying (5.7) and consider $\Phi := \begin{pmatrix} \varphi \\ \partial_t \varphi \end{pmatrix}$.

As it was firstly observed in [11], the existence of $(\varphi_{A_1,\ldots,A_N})_{(A_1,\ldots,A_N)\in\mathbb{R}^N}$ verifying (5.2) in Theorem 5.2 is a consequence of the following crucial

Proposition 5.12. *Let $j \in \{1, \ldots, N\}$ and $A_j \in \mathbb{R}$. Then there exist $t_0 > 0$, $C > 0$, and a solution u of* (NLKG)*, defined on* $[t_0, +\infty)$ *such that*

$$\forall\, t \geq t_0, \qquad \|U(t) - \Phi(t) - A_j e^{-e_j t} Y_{+,j}(t)\|_{H^1 \times L^2} \leq C e^{-(e_j + \sigma)t},$$

where $U := \begin{pmatrix} u \\ \partial_t u \end{pmatrix}$.

By (5.7), assuming that the preceding proposition holds, and considering $A_1, \ldots, A_N \in \mathbb{R}$, we indeed obtain a solution ϕ_{A_1} of (NLKG) and $t_1 > 0$ such that

$$\forall\, t \geq t_1, \qquad \|\Phi_{A_1}(t) - \Phi_0(t) - A_1 e^{-e_1 t} Y_{+,1}(t)\|_{H^1 \times L^2} \leq C e^{-(e_1 + \sigma)t}$$

with obvious notations. We notice that ϕ_{A_1} is a multi-soliton. Now, assume that we have constructed, for some $j \in \{1, \ldots, N-1\}$, a family of multi-solitons $\phi_{A_1}, \ldots, \phi_{A_1,\ldots,A_j}$ such that there exists $t_j > 0$ such that for all $k = 1, \ldots, j$,

$$\forall\, t \geq t_j, \qquad \|\Phi_{A_1,\ldots,A_k}(t) - \Phi_{A_1,\ldots,A_{k-1}}(t) - A_k e^{-e_k t} Y_{+,k}(t)\|_{H^1 \times L^2} \leq C e^{-(e_k + \sigma)t},$$

where $\Phi_{A_1,\dots,A_{k-1}} = \Phi_0$ if $k = 1$. Hence we can apply Proposition 5.12 with $\Phi_{A_1,\dots,A_j}$ instead of Φ and there exist $\Phi_{A_1,\dots,A_{j+1}}$ and $t_{j+1} > 0$ such that

$$\forall\, t \geq t_{j+1}, \qquad \|\Phi_{A_1,\dots,A_{j+1}}(t) - \Phi_{A_1,\dots,A_j}(t) - A_{j+1}e^{-e_{j+1}t}Y_{+,j+1}(t)\|_{H^1\times L^2} \leq Ce^{-(e_{j+1}+\sigma)t}.$$

Thus, by induction on j, we obtain a family of multi-solitons $\phi_{A_1}, \dots, \phi_{A_1,\dots,A_j}$ such that for all $t \geq t_0 := \max\{t_j | j = 1, \dots, N\}$,

$$\begin{aligned}\left\|\Phi_{A_1,\dots,A_N}(t) - \Phi_0(t) - \sum_{j=1}^{N} A_j e^{-e_j t}Y_{+,j}(t)\right\|_{H^1\times L^2} &\leq \sum_{j=1}^{N} \|\Phi_{A_1,\dots,A_j}(t) - \Phi_{A_1,\dots,A_{j-1}}(t) - A_j e^{-e_j t}Y_{+,j}(t)\|_{H^1\times L^2} \\ &\leq C\sum_{j=1}^{N} e^{-(e_j+\sigma)t}.\end{aligned}$$

At this stage, we conclude to (5.2) in Theorem 5.2, using once more (5.7) and the triangular inequality.

Let us already justify also that for all $(A_1, \dots, A_N) \neq (A'_1, \dots, A'_N)$, we have $\varphi_{A_1,\dots,A_N} \neq \varphi_{A'_1,\dots,A'_N}$.

Assume for the sake of contradiction that $\varphi_{A_1,\dots,A_N} = \varphi_{A'_1,\dots,A'_N}$ for some N-uples $(A_1, \dots, A_N) \neq (A'_1, \dots, A'_N)$. Then, we denote

$$i_0 := \min\{i \in \{1, \dots, N\} |\ A'_i \neq A_i\}.$$

From the construction of $\varphi_{A_1,\dots,A_N}$, there exists $C > 0$ such that for t large

$$\left\|\Phi_{A_1,\dots,A_N}(t) - \Phi_{A_1,\dots,A_{i_0-1}}(t) - \sum_{j=i_0}^{N} A_j e^{-e_j t}Y_{+,j}(t)\right\|_{H^1\times L^2} \leq C\sum_{j=i_0}^{N} e^{-(e_j+\sigma)t}. \tag{5.8}$$

Similarly there exists $C' > 0$ such that for t large

$$\left\|\Phi_{A'_1,\dots,A'_N}(t) - \Phi_{A'_1,\dots,A'_{i_0-1}}(t) - \sum_{j=i_0}^{N} A'_j e^{-e_j t}Y_{+,j}(t)\right\|_{H^1\times L^2} \leq C'\sum_{j=i_0}^{N} e^{-(e_j+\sigma)t}. \tag{5.9}$$

Using that $\Phi_{A_1,\dots,A_N}(t) = \Phi_{A'_1,\dots,A'_N}(t)$ and $\Phi_{A_1,\dots,A_{i_0-1}}(t) = \Phi_{A'_1,\dots,A'_{i_0-1}}(t)$, we deduce from (5.8) and (5.9) that for all t sufficiently large

$$e^{-e_{i_0}t}|A_{i_0} - A'_{i_0}| \leq Ce^{-(e_{i_0}+\sigma)t}.$$

Hence, letting $t \longrightarrow +\infty$, we obtain $A_{i_0} - A'_{i_0} = 0$, which leads to a contradiction.
This ends the proof of Theorem 5.2.

5.2.1 Compactness argument assuming uniform estimate

The goal of this subsection is to explain how to prove Proposition 5.12; for this, we follow the strategy of Combet [11] and Côte and Muñoz [20], both inspired from pioneering work by Martel [63] and Côte, Martel and Merle [19]. One key ingredient in the construction is the obtainment of uniform estimates satisfied by a sequence of approximating solutions of (NLKG).

We fix $j \in \{1, \dots, N\}$ and $A_j \in \mathbb{R}$. Let $(S_n)_n$ be an increasing sequence of time such that $S_n \to +\infty$. Let us consider $\mathfrak{b}_n = (b_{n,k})_{j<k\leq N} \in \mathbb{R}^{N-j}$ the generic term of a sequence of parameters to be determined, and let u_n be the maximal solution of (NLKG) such that

$$U_n(S_n) = \Phi(S_n) + A_j e^{-e_j S_n} Y_{+,j}(S_n) + \sum_{k>j} b_{n,k} Y_{+,k}(S_n), \tag{5.10}$$

where $U_n := \begin{pmatrix} u_n \\ \partial_t u_n \end{pmatrix}$.

Concerning u_n, we claim:

Proposition 5.13. *There exist $n_0 \geq 0$ and $t_0 > 0$ (independent of n) such that for each $n \geq n_0$, there exists $\mathfrak{b}_n \in \mathbb{R}^{N-j}$ with $|\mathfrak{b}_n| \leq 2e^{-(e_j+2\sigma)t}$ and such that U_n is defined on $[t_0, S_n]$ and satisfies*

$$\forall\, t \in [t_0, S_n], \qquad \|U_n(t) - \Phi(t) - A_j e^{-e_j t} Y_{+,j}(t)\|_{H^1\times L^2} \leq Ce^{-(e_j+\sigma)t}. \tag{5.11}$$

The b_n take the role of modulation parameters and are to be determined (if indeed possible) so that U_n fulfills (5.11), thus is a natural candidate in order to "approximate" the desired solution U which is the object of Proposition 5.13.

We postpone the proof of the previous statement at the next subsection; for the time being, let us assume that Proposition 5.13 is satisfied and let us show how it implies Proposition 5.12. In fact, the existence of U is due to the continuity of the flow of (NLKG) for the weak $H^1 \times L^2$ topology. We explicit the construction of U below, following the same strategy as [19, paragraph 2.2, step 2] or [20, section 4].

Proof of Proposition 5.12. We observe that the sequence $(\|U_n(t_0)\|_{H^1\times L^2})_{n\in\mathbb{N}}$ is bounded; thus there exist a subsequence of $(U_n(t_0))_{n\in\mathbb{N}}$, say $(U_{n_k}(t_0))_{k\in\mathbb{N}}$, and $U_0 \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$ such that $(U_{n_k}(t_0))_{k\in\mathbb{N}}$ converges to U_0 in the sense of the weak topology in $H^1(\mathbb{R})\times L^2(\mathbb{R})$. Let us consider U, defined as the maximal solution of (NLKG) such that $U(t_0) = U_0$.
Let $t \geq t_0$. For k sufficiently large, $S_{n_k} \geq t$ and thus U_{n_k} is defined on $[t_0, t]$. By a standard result (we refer to [20, Lemma 10] and [104, Theorem 1.2]), U is defined on $[t_0, t]$ and $(U_{n_k}(t))_k$ converges weakly to $U(t)$ in $H^1(\mathbb{R}) \times L^2(\mathbb{R})$.
Moreover, by property of the weak limit,

$$\begin{aligned} \|U(t) - \Phi(t) - A_j e^{-e_j t} Y_{+,j}(t)\|_{H^1\times L^2} &\leq \liminf_{k\to+\infty} \|U_{n_k}(t) - \Phi(t) - A_j e^{-e_j t} Y_{+,j}(t)\|_{H^1\times L^2} \\ &\leq C_0 e^{-(e_j+\sigma)t}. \end{aligned}$$

□

Now, the remainder of Section 5.2 is devoted to the proof of Proposition 5.13.

5.2.2 Proof of Proposition 5.13

For ease of reading, we will drop the index n for the rest of this subsection (except for S_n), that is, we will write U for U_n, $\mathfrak{b}$ for $\mathfrak{b}_n$, etc.

Let us introduce the following variable (which depends on n)

$$W(t) := U(t) - \Phi(t) - A_j e^{-e_j t} Y_{+,j}(t)$$

and for all $k \in \{1, \dots, N\}$,

$$\alpha_{\pm,k}(t) := \langle W(t), Z_{\pm,k}(t) \rangle$$

(which depends on $\mathfrak{b}$ in particular by definition of $U = U_n$ (5.10)).
We denote also $\alpha_-(t) := (\alpha_{-,k}(t))_{j<k\leq N}$.

Modulated final data and strategy of the proof of Proposition 5.13

We make the first step in order to determine the appropriate modulation parameter $\mathfrak{b}$. We obtain $\mathfrak{b}$ as the solution of a well-chosen equation; this is the object of the following

Lemma 5.2. *There exists $n_0 \geq 0$ such that for all $n \geq n_0$ and for all $\mathfrak{a} \in \mathbb{R}^{N-j}$, there exists a unique $\mathfrak{b} \in \mathbb{R}^{N-j}$ such that $\|\mathfrak{b}\| \leq 2\|\mathfrak{a}\|$ and $\alpha_-(S_n) = \mathfrak{a}$.*

Proof. Let us consider the linear application

$$\begin{array}{cccc} \Psi : & \mathbb{R}^{N-j} & \longrightarrow & \mathbb{R}^{N-j} \\ & \mathfrak{b} = (b_l)_{j<l\leq N} & \mapsto & \left(\sum_{l>j} b_l \langle Y_{+,l}(S_n), Z_{-,l}(S_n) \rangle \right)_{j<k\leq N} \end{array}.$$

Its matrix in the canonical basis of $\mathbb{R}^{N-j}$ has generic entry $\psi_{k,l} := \langle Y_{+,j+l}(S_n), Z_{-,j+k}(S_n) \rangle$ where $(k, l) \in \{1, \dots, N\}^2$.
Since $\psi_{k,l} = 1$ if $k = l$ and $|\psi_{k,l}| \leq C_0 e^{-\sigma S_n}$ for $k \neq l$, with $C_0 > 0$ independent of n, we have $\Psi = Id + M$ with $\|M\| \leq \frac{1}{2}$ for large values of n. Thus Ψ is invertible (for n large) and $\|\Psi^{-1}\| \leq 2$. We deduce the content of Lemma 5.2 by taking n_0 large enough and by considering, for a given $\mathfrak{a} \in \mathbb{R}^{N-j}$, the element $\mathfrak{b} := \Psi^{-1}(\mathfrak{a})$. □

Roughly speaking, Lemma 5.2 reflects that estimate (5.11) is to be proven by choosing a relevant vector $\mathfrak{a} = a_-(S_n)$.
The reason why we determine $\mathfrak{b}$ according to the value of $\alpha_-(S_n)$ essentially comes from the directions $Z_{-,k}$, which yield "instability" in some sense (given Claim 5.16 below), and also from definition (5.12) below.

At this stage, we notice that we already have:

Claim 5.14. *We have:*

1. $\forall\, k \in \{1, \dots, N\}, \quad |\alpha_{+,k}(S_n)| \leq C|\mathfrak{b}|e^{-2\sigma S_n}$.
2. $\forall\, k \in \{1, \dots, j\}, \quad |\alpha_{-,k}(S_n)| \leq C|\mathfrak{b}|e^{-2\sigma S_n}$.
3. $\|W(S_n)\|_{H^1 \times L^2} \leq C|\mathfrak{b}|$.

Let $t_0 > 0$ independent of n to be chosen later and $\mathfrak{a} \in B_{\mathbb{R}^{N-j}}(e^{-(e_j+2\sigma)S_n})$ to be determined. We consider the associated data $\mathfrak{b}$ given by Lemma 5.2 and U defined in (5.10).
Let us define

$$T(\mathfrak{a}) := \inf\{T \geq t_0 \mid \forall\, t \in [T, S_n],\ \|W(t)\|_{H^1\times L^2} \leq e^{-(e_j+\sigma)t} \text{ and } e^{(e_j+2\sigma)t}\alpha_-(t) \in B_{\mathbb{R}^{N-j}}(1)\}. \tag{5.12}$$

We observe that Proposition 5.13 holds if for all n, we can find $\mathfrak{a}$ such that $T(\mathfrak{a}) = t_0$. In the rest of the proof, our goal is thus to prove the existence of such an element $\mathfrak{a}$.

To this end, we will first of all improve the estimate on $\|W(t)\|_{H^1\times L^2}$ which falls within the definition of $T(\mathfrak{a})$. This is the object of the following paragraph. Then, we will only need to care about the second condition, which implies a control of $\alpha_-(t)$.

Improvement of the estimate on $\|W\|_{H^1\times L^2}$

For notation purposes and ease of reading, we sometimes omit the index n and also write $\mathrm{O}\,(G(t))$ in order to refer to a function g which a priori depends on n and such that there exists $C \geq 0$ (independent of n) such that for all n large and for all $t \in [t_n^*, S_n]$, $|g(t)| \leq C|G(t)|$.

Lemma 5.3. *There exists $K_0 > 0$ such that for all $t \in [T(\mathfrak{a}), S_n]$,*

$$\|W(t)\|_{H^1\times L^2} \leq \frac{K_0}{t^{\frac{1}{4}}} e^{-(e_j+\sigma)t}.$$

The whole subsection consists of the proof of this lemma.

Step 1: Estimates on $\alpha_{\pm,k}$ Let us begin with the computation of the time derivative of W.

Claim 5.15. *We have for all $k \in \{1, \dots, N\}$,*

$$\begin{aligned}\partial_t W = &\begin{pmatrix} 0 & Id \\ \partial_x^2 - Id + f'(\varphi) & 0 \end{pmatrix} W + A_j e^{-e_j t}\left[\begin{pmatrix} \beta_j\partial_x & Id \\ \partial_x^2 - Id + f'(Q_k) & \beta_j\partial_x \end{pmatrix} Y_{+,j} + e_j Y_{+,j}\right] \\ &+ A_j e^{-e_j t}\begin{pmatrix} 0 & 0 \\ f'(\phi) - f'(Q_k) & 0 \end{pmatrix} Y_{+,j} + \begin{pmatrix} 0 \\ g \end{pmatrix},\end{aligned} \tag{5.13}$$

where $g := f(u) - f(\varphi) - f'(\varphi)(u-\varphi)$ satisfies

$$\|g(t)\|_{L^\infty} = \mathrm{O}\left(\|u-\varphi\|_{H^1}^2\right).$$

Proof. Claim 5.15 follows from the fact that both U and Φ satisfy (NLKG') and is also a consequence of the following Taylor inequality (f is $\mathscr{C}^2$)

$$|f(u) - f(\varphi) - f'(\varphi)(u-\varphi)|(t) \leq C\|u(t) - \varphi(t)\|_{L^\infty}^2$$

and the Sobolev embedding $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$. □

Now, we are in a position to prove the following estimate on $\alpha_{\pm,k}$.

Claim 5.16. *For all $k \in \{1, \dots, N\}$ and for all $t \in [T(\mathfrak{a}), S_n]$, we have*

$$\left| \frac{d}{dt}\alpha_{\pm,k}(t) \mp e_k \alpha_{\pm,k}(t) \right| \leq C \left(e^{-4\sigma t} \|W(t)\|_{H^1 \times L^2} + \|W(t)\|^2_{H^1 \times L^2} + e^{-(e_j + 4\sigma)t} \right). \tag{5.14}$$

Proof. Let $k \in \{1, \dots, N\}$. By means of (5.13) and since $\partial_t Z_{\pm,k} = -\beta_k \partial_x Z_{\pm,k}$, we compute

$$\begin{aligned}
\frac{d}{dt}\alpha_{\pm,k}(t) &= \langle \partial_t W, Z_{\pm,k} \rangle + \langle W, \partial_t Z_{\pm,k} \rangle \\
&= \left\langle W, \begin{pmatrix} -\beta_k \partial_x & \partial_x^2 - Id + f'(\varphi) \\ Id & -\beta_k \partial_x \end{pmatrix} Z_{\pm,k} \right\rangle \\
&\quad + A_j e^{-e_j t} \left\langle Y_{+,j}, \begin{pmatrix} -\beta_j \partial_x & \partial_x^2 - Id + f'(Q_k) \\ Id & -\beta_j \partial_x \end{pmatrix} Z_{\pm,k} \right\rangle \\
&\quad + A_j e^{-e_j t} \left[\left\langle Y_{+,j}, \begin{pmatrix} 0 & f'(Q_k) - f'(\varphi) \\ 0 & 0 \end{pmatrix} Z_{\pm,k} \right\rangle + e_j \langle Y_{+,j}, Z_{\pm,k} \rangle \right] \\
&\quad + \mathrm{O}\left(\|U - \Phi\|^2_{H^1 \times L^2} \right).
\end{aligned}$$

Let us notice first that

$$\left\langle W, \begin{pmatrix} -\beta_k \partial_x & \partial_x^2 - Id + f'(\varphi) \\ Id & -\beta_k \partial_x \end{pmatrix} Z_{\pm,k} \right\rangle = \langle W, \mathscr{H}_k Z_{\pm,k} \rangle + \left\langle W, \begin{pmatrix} 0 & f'(\varphi) - f'(Q_k) \\ 0 & 0 \end{pmatrix} Z_{\pm,k} \right\rangle.$$

We have

$$\langle W, \mathscr{H}_k Z_{\pm,k} \rangle = \langle W, \pm e_k Z_{\pm,k} \rangle = \pm e_k \alpha_{\pm,k}$$

and

$$\begin{aligned}
&\left| \left\langle W, \begin{pmatrix} 0 & f'(\varphi) - f'(Q_k) \\ 0 & 0 \end{pmatrix} Z_{\pm,k} \right\rangle \right| \\
&\quad \leq \|W\|_{H^1 \times L^2} \| \left(f'(\varphi) - f'(Q_k) \right) Z_{\pm,k} \|_{H^1 \times L^2} \\
&\quad \leq C \left\| \varphi - \sum_{i=1}^N Q_i \right\|_{L^\infty} \|W\|_{H^1 \times L^2} + C \|W\|_{H^1 \times L^2} \sum_{i \neq k} \|Q_i Z_{\pm,k}\|_{H^1 \times L^2} \\
&\quad \leq C e^{-4\sigma t} \|W\|_{H^1 \times L^2}.
\end{aligned}$$

Similarly, we have

$$\left| \left\langle Y_{+,j}, \begin{pmatrix} 0 & f'(Q_k) - f'(\varphi) \\ 0 & 0 \end{pmatrix} Z_{\pm,k} \right\rangle \right| \leq C e^{-4\sigma t}$$

and

$$\begin{aligned}
\left\langle Y_{+,j}, \begin{pmatrix} -\beta_j \partial_x & \partial_x^2 - Id + f'(Q_k) \\ Id & -\beta_j \partial_x \end{pmatrix} Z_{\pm,k} \right\rangle &= \langle Y_{+,j}, \mathscr{H}_k Z_{\pm,k} \rangle + (\beta_k - \beta_j) \langle Y_{+,j}, \partial_x Z_{\pm,k} \rangle \\
&= \pm \langle Y_{+,j}, e_k Z_{\pm,k} \rangle + \mathrm{O}\left(e^{-4\sigma t} \right).
\end{aligned}$$

Indeed, we notice that

$$(\beta_k - \beta_j) \langle Y_{+,j}, \partial_x Z_{\pm,k} \rangle = \begin{cases} 0 & \text{if } k = j \\ \mathrm{O}(e^{-4\sigma t}) & \text{if } k \neq j. \end{cases}$$

Hence, we obtain

$$\begin{aligned}\frac{d}{dt}\alpha_{\pm,k}(t) = &\pm e_k\alpha_{\pm,k} + \mathrm{O}\left(e^{-4\sigma t}\|W\|_{H^1\times L^2}\right)\\ &+ A_j e^{-e_j t}\left[\pm\langle Y_{+,j}, e_k Z_{\pm,k}\rangle + e_j\langle Y_{+,j}, Z_{\pm,k} + \mathrm{O}\left(e^{-4\sigma t} + \|U-\Phi\|^2_{H^1\times L^2}\right)\right].\end{aligned} \tag{5.15}$$

Now, we observe that

$$\pm e_k\langle Y_{+,j}, Z_{\pm,k}\rangle + e_j\langle Y_{+,j}, Z_{\pm,k}\rangle = \mathrm{O}(e^{-4\sigma t}).$$

This is clear if $k \neq j$ and for $k = j$, we have

$$\pm e_j\langle Y_{+,j}, Z_{\pm,j}\rangle + e_j\langle Y_{+,j}, Z_{\pm,j}\rangle = \begin{cases} 0+0=0 & \text{if } \pm = + \\ -e_j + e_j = 0 & \text{if } \pm = - \end{cases};$$

indeed, we recall from Proposition 5.9

$$\langle Y_{+,j}, Z_{+,j}\rangle = 0 \quad \text{and} \quad \langle Y_{+,j}, Z_{-,j}\rangle = 1.$$

In addition, we have by the well-known inequality $(a+b)^2 \le 2(a^2+b^2)$,

$$\|U-\Phi\|^2_{H^1\times L^2} \le C\left(\|W\|^2_{H^1\times L^2} + e^{-2e_j t}\right).$$

Considering that $2e_j \ge e_j + 4\sigma$, we have thus finished the proof of the claim. □

Step 2: Control of the stable directions

Claim 5.17. *We have for all $k \in \{1,\dots,N\}$, for all $t \in [T(\mathfrak{a}), S_n]$,*

$$|\alpha_{+,k}(t)| \le Ce^{-(e_j+4\sigma)t}. \tag{5.16}$$

Proof. Due to Claim 5.16 and (5.12), we obtain

$$\left|\frac{d}{dt}\alpha_{+,k}(t) - e_k\alpha_{+,k}(t)\right| \le Ce^{-(e_j+4\sigma)t},$$

that is, for all $t \in [T(\mathfrak{a}), S_n]$,

$$\left|\left(e^{-e_k t}\alpha_{+,k}(t)\right)'\right| \le Ce^{-(e_j+e_k+4\sigma)t}.$$

Integrating, we deduce that for all $t \in [T(\mathfrak{a}), S_n]$,

$$|e^{-e_k S_n}\alpha_{+,k}(S_n) - e^{-e_k t}\alpha_{+,k}(t)| \le Ce^{-(e_j+e_k+4\sigma)t}.$$

Thus,

$$|\alpha_{+,k}(t)| \le |\alpha_{+,k}(S_n)| + Ce^{-(e_j+4\sigma t)}.$$

From Claim 5.14 and Lemma 5.2, we have

$$\begin{aligned}|\alpha_{+,k}(S_n)| &\le Ce^{-2\sigma S_n}|\mathfrak{b}|\\ &\le Ce^{-2\sigma S_n}e^{-(e_j+2\sigma)S_n}\\ &\le Ce^{-(e_j+4\sigma)t}.\end{aligned}$$

Consequently, Claim 5.17 indeed holds. □

Step 3: Control of the unstable directions for $k \leq j$

Claim 5.18. *We have for all $k \in \{1, \ldots, j\}$, for all $t \in [T(\mathfrak{a}), S_n]$,*

$$|\alpha_{-,k}(t)| \leq Ce^{-(e_j+4\sigma)t}. \tag{5.17}$$

Proof. As in the preceding step, we have for all $k \in \{1, \ldots, N\}$ and $t \in [T(\mathfrak{a}), S_n]$,

$$\left|\frac{d}{dt}\alpha_{-,k}(t) + e_k\alpha_{-,k}(t)\right| \leq Ce^{-(e_j+4\sigma)t}, \tag{5.18}$$

which writes also

$$\left|\left(e^{e_k t}\alpha_{-,k}(t)\right)'\right| \leq Ce^{(e_k-e_j+4\sigma)t}.$$

For $k \leq j$, we have $e_k \leq e_j$, and so by integration, we obtain

$$|\alpha_{-,k}(t)| \leq e^{e_k(S_n-t)}|\alpha_{-,k}(S_n)| + Ce^{-(e_j+4\sigma)t}.$$

But again from Claim 5.14 and Lemma 5.2, we infer

$$\begin{aligned} e^{e_k(S_n-t)}|\alpha_{-,k}(S_n)| &\leq Ce^{e_k(S_n-t)}e^{-2\gamma S_n}e^{-(e_j+2\sigma)S_n} \\ &\leq Ce^{(S_n-t)(e_k-e_j)}e^{-e_j t}e^{-4\sigma S_n} \\ &\leq Ce^{-(e_j+4\sigma)t}. \end{aligned}$$

Thus

$$\forall\, k \in \{1, \ldots, j\},\ \forall\, t \in [T(\mathfrak{a}), S_n], \qquad |\alpha_{-,k}(t)| \leq Ce^{-(e_j+4\sigma)t}.$$

□

Step 4: Control of a Lyapunov functional satisfying a coercivity property Let us consider

$$\begin{aligned} \psi :\ & \mathbb{R} \to \mathbb{R} \\ & x \mapsto \tfrac{2}{\pi}\operatorname{Arctan}\left(e^{-x}\right). \end{aligned}$$

We define for all $k = 1, \ldots, N-1$,

$$\psi_k(t,x) := \psi\left(\frac{1}{\sqrt{t}}\left(x - \frac{\beta_{\eta(k)} + \beta_{\eta(k+1)}}{2}t - \frac{x_{\eta(k)} + x_{\eta(k+1)}}{2}\right)\right),$$

and then

$$\begin{aligned} \phi_1(t) &= \psi_1(t) \\ \phi_k(t) &= \psi_k(t) - \psi_{k-1}(t) \text{ for all } k = 2, \ldots, N-1, \\ \phi_N(t) &= 1 - \psi_{N-1}(t). \end{aligned}$$

Recall that the permutation η has been chosen so that $-1 < \beta_{\eta(1)} < \cdots < \beta_{\eta(N)} < 1$.

Now, let us introduce for all $k \in \{1, \ldots, N\}$

$$\mathcal{F}_{W,k}(t) = \int_{\mathbb{R}} \left(w_1^2 + (\partial_x w_1)^2 + w_2^2 - f'(Q_{\eta(k)})w_1^2 + 2\beta_{\eta(k)}\partial_x w_1 w_2\right)\phi_k\, dx,$$

and

$$\mathcal{F}_W(t) := \sum_{k=1}^{N} \mathcal{F}_{W,k}(t).$$

By means of Proposition 5.10 and a usual localization argument [80], we obtain that $\mathcal{F}_W$ is coercive on a subspace of $H^1 \times L^2$ of finite codimension. More precisely, there exists $\mu > 0$ such that

$$\mathcal{F}_W(t) \geq \mu \|W(t)\|^2_{H^1\times L^2} - \frac{1}{\mu}\sum_{k=1}^{N}\left(\langle W, \partial_x R_k\rangle^2 + \langle W, Z_{+,k}\rangle^2 + \langle W, Z_{-,k}\rangle^2\right). \tag{5.19}$$

We state the following control about the derivative of $\mathcal{F}_W$:

Claim 5.19. *For t_0 large and for all $t \in [T(\mathfrak{a}), S_n]$,*

$$\left|\frac{d}{dt}\mathcal{F}_W(t)\right| \leq \frac{C}{\sqrt{t}}\|W\|^2_{H^1\times L^2}. \tag{5.20}$$

Proof. Let us rewrite $\mathcal{F}_{W,k}$ differently, using the notations developed in the introduction. Relying on integrations by parts, our computations lead to:

$$\langle (H_{\eta(k)}W)\phi_k, W\rangle = \mathcal{F}_{W,k}(t) - \frac{1}{2}\int_{\mathbb{R}} w_1^2 \partial_x^2\phi_k \, dx + \beta_{\eta(k)}\int_{\mathbb{R}} w_1 w_2 \partial_x \phi_k \, dx.$$

Thus

$$\mathcal{F}_{W,k}(t) = \langle (H_{\eta(k)}W)\phi_k, W\rangle + \mathrm{O}\left(\frac{1}{\sqrt{t}}\|W\|^2_{H^1\times L^2}\right). \tag{5.21}$$

We immediately have

$$\frac{d}{dt}\langle (H_{\eta(k)}W)\phi_k, W\rangle = \langle (H_{\eta(k)}W)\phi_k, \partial_t W\rangle + \langle \partial_t(H_{\eta(k)}W)\phi_k, W\rangle + \langle (H_{\eta(k)}W)\partial_t\phi_k, W\rangle.$$

Besides

$$\begin{aligned}
&\langle \partial_t(H_{\eta(k)}W)\phi_k, W\rangle + \langle (H_{\eta(k)}W)\partial_t\phi_k, W\rangle \\
&\quad = \langle (H_{\eta(k)}\partial_t W)\phi_k, W\rangle + \beta_{\eta(k)}\int_{\mathbb{R}} \partial_x Q_{\eta(k)} f''(Q_{\eta(k)}) w_1^2 \phi_k \, dx + \mathrm{O}\left(\frac{1}{\sqrt{t}}\|W\|^2_{H^1\times L^2}\right) \\
&\quad = \langle H_{\eta(k)}(\partial_t W), W\phi_k\rangle + \beta_{\eta(k)}\int_{\mathbb{R}} \partial_x Q_{\eta(k)} f''(Q_{\eta(k)}) w_1^2 \phi_k \, dx + \mathrm{O}\left(\frac{1}{\sqrt{t}}\|W\|^2_{H^1\times L^2}\right).
\end{aligned}$$

Since $H_{\eta(k)}$ is a self-adjoint operator, we have

$$\langle H_{\eta(k)}(\partial_t W), W\phi_k\rangle = \langle \partial_t W, H_{\eta(k)}(W\phi_k)\rangle.$$

By a straightforward calculation, we have moreover

$$\langle H_{\eta(k)}(W\phi_k), \partial_t W\rangle = \langle (H_{\eta(k)}W)\phi_k, \partial_t W\rangle + \mathrm{O}\left(\frac{1}{\sqrt{t}}\|W\|^2_{H^1\times L^2}\right).$$

At this stage, we thus obtain

$$\frac{d}{dt}\langle (H_{\eta(k)}W)\phi_k, W\rangle = 2\langle (H_{\eta(k)}W)\phi_k, \partial_t W\rangle + \beta_{\eta(k)}\int_{\mathbb{R}} \partial_x Q_{\eta(k)} f''(Q_{\eta(k)}) w_1^2 \phi_k \; dx + \mathrm{O}\left(\frac{1}{\sqrt{t}}\|W\|^2_{H^1\times L^2}\right).$$

Now, by (5.13), we write

$$\left\langle (H_{\eta(k)}W)\phi_k, \partial_t W\right\rangle = I_1 + I_2 + I_3$$

where

$$I_1 := \left\langle \begin{pmatrix} T_{\eta(k)} & 0 \\ 0 & Id \end{pmatrix} W\phi_k, \begin{pmatrix} 0 & Id \\ \partial_x^2 - Id + f'(\varphi) & 0 \end{pmatrix} W \right\rangle$$

by denoting $T_i = -\partial_x^2 + Id - f'(Q_i)$ for all $i = 1, \dots, N$,

$$I_2 := \beta_{\eta(k)} \left\langle \begin{pmatrix} 0 & -\partial_x \\ \partial_x & 0 \end{pmatrix} W\phi_k, \begin{pmatrix} 0 & Id \\ -T_{\eta(k)} + f'(\varphi) - f'(Q_{\eta(k)}) & 0 \end{pmatrix} W \right\rangle,$$

and

$$I_3 := A_j e^{-e_j t} \left\langle (H_{\eta(k)}W)\phi_k, \begin{pmatrix} \beta_j\partial_x & Id \\ \partial_x^2 - Id + f'(Q_j) & \beta_j\partial_x \end{pmatrix} Y_{+,j} \right\rangle + \left\langle (H_{\eta(k)}W)\phi_k, e_j Y_{+,j} + \begin{pmatrix} 0 & 0 \\ f'(\varphi) - f'(Q_j) & 0 \end{pmatrix} Y_{+,j} \right\rangle.$$

Let us deal with I_1: we observe that

$$\begin{aligned}
I_1 &= \left\langle \begin{pmatrix} T_{\eta(k)} & 0 \\ 0 & Id \end{pmatrix} W\phi_k, \begin{pmatrix} 0 & Id \\ -T_{\eta(k)} + f'(\varphi) - f'(Q_{\eta(k)}) & 0 \end{pmatrix} W \right\rangle \\
&= \left\langle W, \begin{pmatrix} T_{\eta(k)} & 0 \\ 0 & Id \end{pmatrix} \begin{pmatrix} 0 & Id \\ -T_{\eta(k)} & 0 \end{pmatrix} W\phi_k \right\rangle + \mathrm{O}\left(e^{-4\sigma t}\|W\|^2_{H^1\times L^2}\right) \\
&= \left\langle W, \begin{pmatrix} 0 & T_{\eta(k)} \\ -T_{\eta(k)} & 0 \end{pmatrix} W\phi_k \right\rangle + \mathrm{O}\left(e^{-4\sigma t}\|W\|^2_{H^1\times L^2}\right) \\
&= -\int_{\mathbb{R}} w_1 \partial_x^2 w_2 \phi_k \; dx + \int_{\mathbb{R}} w_2 \partial_x^2 w_1 \phi_k \; dx + \mathrm{O}\left(\frac{1}{\sqrt{t}}\|W\|^2_{H^1\times L^2}\right) \\
&= \mathrm{O}\left(\frac{1}{\sqrt{t}}\|W\|^2_{H^1\times L^2}\right).
\end{aligned}$$

We have

$$\begin{aligned}
I_2 &= \beta_{\eta(k)} \left\langle \begin{pmatrix} 0 & -\partial_x \\ \partial_x & 0 \end{pmatrix} W\phi_k, \begin{pmatrix} 0 & Id \\ -T_{\eta(k)} & 0 \end{pmatrix} W \right\rangle + \mathrm{O}\left(e^{-(e_j+4\sigma)t}\|W\|_{H^1\times L^2}\right) \\
&= -\beta_{\eta(k)} \int_{\mathbb{R}} \partial_x w_2 w_2 \phi_k \; dx + \beta_{\eta(k)} \int_{\mathbb{R}} \partial_x w_1 (\partial_x^2 w_1 - w_1 + f'(Q_{\eta(k)}) w_1)\phi_k \; dx \\
&\quad + \mathrm{O}\left(e^{-(e_j+4\sigma)t}\|W\|_{H^1\times L^2}\right)
\end{aligned}$$

$$= -\frac{\beta_{\eta(k)}}{2}\int_{\mathbb{R}} w_1^2 \partial_x Q_{\beta_{\eta(k)}} f''(Q_{\beta_{\eta(k)}})\phi_k \, dx + \mathrm{O}\left(\frac{1}{\sqrt{t}}\|W\|^2_{H^1\times L^2} + e^{-(e_j+4\sigma)t}\|W\|_{H^1\times L^2}\right).$$

In addition, we have

$$\begin{aligned} I_3 &= A_j e^{-e_j t}\left\langle (H_{\eta(k)}W)\phi_k, JZ_{+,j} + e_j Y_{+,j}\right\rangle + \mathrm{O}\left(e^{-(4\sigma+e_j)t}\|W\|_{H^1\times L^2}\right) \\ &= A_j e^{-e_j t}\left(\left\langle W\phi_k, -\mathscr{H}_{\eta(k)}Z_{+,j}\right\rangle + e_j\langle W\phi_k, H_{\eta(k)}Y_{+,k}\rangle\right) \\ &\quad + \mathrm{O}\left(\frac{1}{\sqrt{t}}\|W\|^2_{H^1\times L^2} + e^{-(4\sigma+e_j)t}\|W\|_{H^1\times L^2}\right) \\ &= \mathrm{O}\left(\frac{1}{\sqrt{t}}\|W\|^2_{H^1\times L^2} + e^{-(4\sigma+e_j)t}\|W\|_{H^1\times L^2}\right). \end{aligned}$$

Note that the last line of the previous equality is a consequence of the following observation: if $\eta(k) = j$, we have $\mathscr{H}_j Z_{+,j} = e_j Z_{+,j}$ and $H_j Y_{+,j} = Z_{+,j}$ so that

$$A_j e^{-e_j t}\left(\left\langle W\phi_k, -\mathscr{H}_{\eta(k)}Z_{+,j}\right\rangle + e_j\langle W\phi_k, H_{\eta(k)}Y_{+,j}\rangle\right) = 0.$$

If $\eta(k) \neq j$, we have

$$A_j e^{-e_j t}\left(\left\langle W\phi_k, -\mathscr{H}_{\eta(k)}Z_{+,j}\right\rangle + e_j\langle W\phi_k, H_{\eta(k)}Y_{+,j}\rangle\right) = \mathrm{O}\left(e^{-(e_j+4\sigma)t}\|W\|_{H^1\times L^2}\right).$$

Gathering the preceding computations yields

$$\left|\frac{d}{dt}\mathcal{F}_{W,k}(t)\right| \leq C\left(\frac{1}{\sqrt{t}}\|W\|^2_{H^1\times L^2} + e^{-(e_j+4\sigma)t}\|W\|_{H^1\times L^2}\right),$$

hence the expected claim, by summing on k. □

<u>**Step 5:**</u> **Control of the directions $\partial_x R_k$** To obtain a control of the scalar products $\langle W, \partial_x R_k\rangle$ which is more precise than the a priori control by $\|W\|_{H^1\times L^2}$, let us introduce the following modulated variable $\tilde{W}$:

$$\tilde{W}(t) = W(t) + \sum_{k=1}^{N} a_k(t)\partial_x R_k(t), \tag{5.22}$$

where $a_k(t) \in \mathbb{R}$, $k = 1, \ldots, N$ are chosen so that for all $l = 1, \ldots, N$, $\langle \tilde{W}(t), \partial_x R_l(t)\rangle = 0$. Existence and uniqueness of the family $(a_k(t))_{k\in\{1,\ldots,N\}}$ are justified by the fact that the (interaction) $N \times N$-matrix with generic entry $\langle \partial_x R_k(t), \partial_x R_l(t)\rangle$ is invertible for t large enough.

Notice that

$$|a_k(t)| \leq C\|W(t)\|_{H^1\times L^2} \leq Ce^{-(e_j+\sigma)t}. \tag{5.23}$$

The functional $\mathcal{F}_{\tilde{W}}(t)$, defined as $\mathcal{F}_W(t)$ by changing W in $\tilde{W}$, satisfies the following coercivity property:

$$\|\tilde{W}\|^2_{H^1\times L^2} \leq C\left(\mathcal{F}_{\tilde{W}}(t) + \sum_{k=1}^{N}\left(\langle \tilde{W}, Z_{+,k}\rangle^2 + \langle \tilde{W}, Z_{-,k}\rangle^2\right)\right). \tag{5.24}$$

We have

$$\mathcal{F}_{\tilde{W}}(t) \leq \mathcal{F}_W(t) + \mathrm{O}\left(e^{-4\sigma t}\|W\|^2_{H^1\times L^2}\right)$$

and we have moreover by Proposition 5.11 and (5.22).

$$\langle \tilde{W}, Z_{\pm,k}\rangle^2 \leq \alpha_{\pm,k}^2 + e^{-2(e_j+5\sigma)t}.$$

Claim 5.20 (Estimate on $\tilde{W}$)**.** *We have*

$$\forall\, t \in [T(\mathfrak{a}), S_n], \qquad \|\tilde{W}(t)\|_{H^1\times L^2}^2 \leq \frac{1}{\sqrt{t}} e^{-(2e_j+2\sigma)t}.$$

Proof. Let t belong to $[T(\mathfrak{a}), S_n]$. We obtain by (5.24) and by integration of (5.20) on $[t, +\infty)$ (which is indeed possible by definition of $T(\mathfrak{a})$) that

$$\|\tilde{W}(t)\|_{H^1\times L^2}^2 \leq \frac{C}{\sqrt{t}} e^{-2(e_j+\sigma)t} + C\sum_{\pm,k} \alpha_{\pm,k}^2 + Ce^{-2(e_j+4\sigma)t}.$$

Using the estimate on $\alpha_{\pm,k}$ provided by the definition of $T(\mathfrak{a})$ and Claim 5.17, we then infer:

$$\|\tilde{W}(t)\|_{H^1\times L^2}^2 \leq \frac{C}{\sqrt{t}} e^{-2(e_j+\sigma)t} + Ce^{-(2e_j+4\sigma)t}.$$

This concludes the proof of the claim. □

Claim 5.21 (Control of the modulation parameters)**.** *We have for all $k = 1, \ldots, N$,*

$$\forall\, t \in [T(\mathfrak{a}), S_n], \qquad |a_k(t)| \leq \frac{C}{t^{\frac{1}{4}}} e^{-(e_j+\sigma)t}.$$

Proof. By definition of the modulation parameters a_k, we have $\langle \tilde{W}, \partial_x R_k\rangle = 0$. Thus, we have by differentiation with respect to t:

$$\langle \partial_t \tilde{W}, \partial_x R_k\rangle + \langle \tilde{W}, \partial_t\partial_x R_k\rangle = 0.$$

By Proposition 5.11, we have for $l \neq k$,

$$\langle \partial_x R_l, \partial_x R_k\rangle = \mathrm{O}\left(e^{-4\sigma t}\right)$$

and for all l,

$$\langle \partial_t\partial_x R_l, \partial_x R_k\rangle = \mathrm{O}\left(e^{-4\sigma t}\right).$$

We deduce that

$$a_k'(t)\langle \partial_x R_k, \partial_x R_k\rangle + \langle \partial_t W, \partial_x R_k\rangle + \langle \tilde{W}, \partial_t\partial_x R_k\rangle = \mathrm{O}\left(e^{-4\sigma t}\|W(t)\|_{H^1\times L^2}\right).$$

We have in addition

$$\left\langle W, \begin{pmatrix} 0 & \partial_x^2 - Id + f'(\varphi) \\ Id & 0 \end{pmatrix} \partial_x R_k \right\rangle + \langle \tilde{W}, \partial_{tx} R_k\rangle = \mathrm{O}\left(\|\tilde{W}\|_{H^1\times L^2}\right).$$

What is more,

$$\left|A_j e^{-e_j t} e_j \langle Y_{+,j}, \partial_x R_k\rangle\right| \leq Ce^{-(e_j+4\sigma)t},$$

again by Proposition 5.11.

Hence,

$$\begin{aligned}|a_k'(t)| &\leq C\|\tilde{W}(t)\|_{H^1\times L^2} + Ce^{-(e_j+4\sigma)t}\\ &\leq \frac{C}{t^{\frac{1}{4}}}e^{-(e_j+\sigma)t} + e^{-(e_j+3\sigma)t}.\end{aligned}$$

□

Now, gathering (5.22), Claim 5.20, and Claim 5.21, we immediately deduce the expected estimate of $\|W\|_{H^1\times L^2}$, which ends the proof of Lemma 5.3.

Control of the unstable directions for $k > j$ and end of the proof

To control $\alpha_- = \left(\alpha_{-,k}\right)_{j<k\leq N}$ and eventually obtain the following statement, we resort to a classical topological argument, already set up in [11] and initially developed by Côte, Martel and Merle [19].

Lemma 5.4. *For t_0 large enough, there exists $\mathfrak{a} \in B_{\mathbb{R}^{N-j}}(e^{-(e_j+2\sigma)S_n})$ such that $T(\mathfrak{a}) = t_0$.*

The proof follows that of Combet [11]. We write it below for the sake of completeness.

Proof. We first choose t_0 sufficiently large such that $\frac{K_0}{\sqrt{t_0}} \leq \frac{1}{2}$. Then, we have by Lemma 5.3

$$\forall\, t \in [T(\mathfrak{a}), S_n], \qquad \|W(t)\|_{H^1\times L^2} \leq \frac{1}{2}e^{-(e_j+\sigma)t}.$$

Assume, for the sake of contradiction, that for all $\mathfrak{a} \in B_{\mathbb{R}^{N-j}}(e^{-(e_j+2\sigma)S_n})$, $T(\mathfrak{a}) > t_0$. As $\|W(T(\mathfrak{a}))\|_{H^1\times L^2} \leq \frac{1}{2}e^{-(e_j+\sigma)T(\mathfrak{a})}$, by definition of $T(\mathfrak{a})$ and continuity of the flow, we have:

$$|\alpha_-(T(\mathfrak{a}))| = 1.$$

(We recall that $\alpha_-(t) = (\alpha_{-,k}(t))_{j<k\leq N}$.) In other words, the map

$$\begin{array}{rccc}\mathcal{M}: & B_{\mathbb{R}^{N-j}}(e^{-(e_j+2\sigma)S_n}) & \longrightarrow & S_{\mathbb{R}^{N-j}}(e^{-(e_j+2\sigma)S_n})\\ & \mathfrak{a} & \longmapsto & e^{-(e_j+2\sigma)(S_n-T(\mathfrak{a}))}\alpha_-(T(\mathfrak{a}))\end{array}$$

is well-defined. Now, we aim at showing that $\mathcal{M}$ is continuous and that its restriction to $S_{\mathbb{R}^{N-j}}(e^{-(e_j+2\gamma)S_n})$ is the identity.

Let $T \in [T_0, T(\mathfrak{a})]$ be such that W is defined on $[T, S_n]$ and, by continuity,

$$\forall\, t \in [T, S_n], \qquad \|W(t)\|_{H^1\times L^2} \leq 1.$$

We consider, for all $t \in [T, S_n]$:

$$\mathcal{N}(t) := \mathcal{N}(\alpha_-(t)) = \left\|e^{(e_j+2\sigma)t}\alpha_-(t)\right\|^2.$$

Claim 5.22. *For t_0 large enough, and for all $t \in [T, S_n]$ such that $\mathcal{N}(t) = 1$, we have:*

$$\mathcal{N}'(t) \leq -(e_{j+1} - e_j - 2\sigma).$$

Proof of Claim 5.22. Let us start from estimate (5.18): for all $k \in \{j+1, \dots, N\}$, for all $t \in [T, S_n]$,

$$\left| \frac{d}{dt}\alpha_{-,k} + e_k \alpha_{-,k} \right| \leq Ce^{-(e_j+4\sigma)t}.$$

Thus we obtain for all $k \in \{j+1, \dots, N\}$,

$$\alpha_{-,k}\frac{d}{dt}\alpha_{-,k} + e_{j+1}\alpha_{-,k}^2 \leq \alpha_{-,k}\frac{d}{dt}\alpha_{-,k} + e_k\alpha_{-,k}^2 \leq Ce^{-(e_j+4\sigma)t}|\alpha_{-,k}|.$$

Then, summing on $k \in \{j+1, \dots, N\}$ leads to

$$\left(|\alpha_-(t)|^2\right)' + 2e_{j+1}|\alpha_-(t)|^2 \leq Ce^{-(e_j+4\sigma)t}|\alpha_-(t)|.$$

Therefore we can estimate:

$$\begin{aligned}\mathcal{N}'(t) &= e^{2(e_j+2\sigma)t}\left[2(e_j+2\sigma)|\alpha_-(t)|^2 + \left(|\alpha_-(t)|^2\right)'\right] \\ &\leq e^{2(e_j+2\sigma)t}\left[2(e_j+2\sigma)|\alpha_-(t)|^2 - 2e_{j+1}|\alpha_-(t)|^2 + Ce^{-(e_j+4\sigma)t}|\alpha_-(t)|\right].\end{aligned}$$

Hence we have for all $t \in |T, S_n]$,

$$\mathcal{N}'(t) \leq -\theta\mathcal{N}(t) + Ce^{e_jt}|\alpha_-(t)|,$$

where $\theta = 2(e_{j+1} - e_j - 2\sigma) > 0$ by definition of σ. In particular, for all $\tau \in [T, S_n]$ satisfying $\mathcal{N}(\tau) = 1$, we have:

$$\mathcal{N}'(\tau) \leq -\theta + Ce^{e_j\tau}|\alpha_-(\tau)| \leq -\theta + Ce^{e_j\tau}e^{-(e_j+2\sigma)\tau} \leq -\theta + Ce^{-2\sigma t_0}.$$

Now, we fix t_0 large enough such that $Ce^{-2\sigma t_0} \leq \frac{\theta}{2}$. Thus for all $\tau \in [T, S_n]$ such that $\mathcal{N}(\tau) = 1$, we have

$$\mathcal{N}'(\tau) \leq -\frac{\theta}{2}.$$

□

Finally, we claim that $\mathfrak{a} \mapsto T(\mathfrak{a})$ is continuous. Indeed, let $\varepsilon > 0$. By definition of $T(\mathfrak{a})$ and by Claim 5.22, there exists $\delta > 0$ such that for all $t \in [T(\mathfrak{a}) + \varepsilon, S_n]$, $\mathcal{N}(t) < 1 - \delta$, and such that $\mathcal{N}(T(\mathfrak{a}) - \varepsilon) > 1 + \delta$. But from continuity of the flow, there exists $\eta > 0$ such that for all $\tilde{\mathfrak{a}}$ satisfying $\|\tilde{\mathfrak{a}} - \mathfrak{a}\| \leq \eta$, we have

$$\forall\, t \in [T(\mathfrak{a}) - \varepsilon, S_n], \qquad |\mathcal{N}(\tilde{\mathfrak{a}}) - \mathcal{N}(\mathfrak{a})| \leq \frac{\delta}{2}.$$

We finally deduce that

$$T(\mathfrak{a}) - \varepsilon \leq T(\tilde{\mathfrak{a}}) \leq T(\mathfrak{a}) + \varepsilon.$$

Hence, $\mathfrak{a} \mapsto T(\mathfrak{a})$ is continuous.

We then obtain that the map $\mathcal{M}$ is continuous. What is more, for $\mathfrak{a} \in S_{\mathbb{R}^{N-j}}(e^{-(e_j+2\sigma)S_n})$, as $\mathcal{N}'(S_n) \leq -(e_{j+1} - e_j - 2\sigma) < 0$, we then deduce by definition of $T(\mathfrak{a})$ that $T(\mathfrak{a}) = S_n$, and thus, $\mathcal{M}(\mathfrak{a}) = \mathfrak{a}$.
The existence of such a map $\mathcal{M}$ contradicts Brouwer's fixed point theorem. Thus, we have finished proving Lemma 5.4.

□

5.3 Classification under condition of the multi-solitons of (NLKG)

Let $N \geq 2$ and $x_1, \ldots, x_N, \beta_1, \ldots, \beta_N$ be $2N$ parameters as in Theorem 5.2. Let U be a solution of (NLKG) such that

$$\left\| U(t) - \sum_{i=1}^{N} R_{\beta_i}(t) \right\|_{H^1 \times L^2} = O\left(\frac{1}{t^\alpha}\right) \qquad \text{as} \quad t \to +\infty \tag{5.25}$$

for some $\alpha > 3$.

The goal of this section is to prove the existence of $A_1, \ldots, A_N \in \mathbb{R}$ such that

$$U = \Phi_{A_1,\ldots,A_N}.$$

Here again, we make the proof for $d = 1$.

We denote by φ a multi-soliton solution associated with these parameters, satisfying (5.7) and $\Phi := \begin{pmatrix} \varphi \\ \partial_t \varphi \end{pmatrix}$. Let us consider $Z := U - \Phi = \begin{pmatrix} z \\ \partial_t z \end{pmatrix}$. Obviously,

$$\|Z(t)\|_{H^1 \times L^2} = O\left(\frac{1}{t^\alpha}\right), \qquad \text{as} \quad t \to +\infty.$$

Our first objective is to improve this comparison, and namely to pass from the polynomial decay to an exponential one.

5.3.1 Exponential convergence to 0 at speed e_1 of $\|Z(t)\|_{H^1 \times L^2}$

Introduction of a new variable by modulation

In a standard way, we modulate the variable Z in order to obtain suitable orthogonality properties, making it possible to obtain crucial estimates when we apply the spectral theory available for (NLKG).

Lemma 5.5. *There exists $t_0 > 0$ and $\mathscr{C}^1$ functions $a_i : [t_0, +\infty) \to \mathbb{R}$ and $b_i : [t_0, +\infty) \to \mathbb{R}$ for all $i = 1, \ldots, N$ such that, defining*

$$E := Z - \sum_{i=1}^{N} a_i \partial_x R_i - \sum_{i=1}^{N} b_i Y_{+,i},$$

we have for all $i = 1, \ldots, N$ and for all $t \geq t_0$:

$$\langle E(t), \partial_x R_i(t) \rangle = 0 \tag{5.26}$$

$$\left\langle E(t), Z_{-,i}(t) \right\rangle = 0. \tag{5.27}$$

Moreover, we have for all $i = 1, \ldots, N$:

$$a_i(t) = \frac{1}{\|\partial_x R_i\|^2} \langle Z(t), \partial_x R_i(t) \rangle + O\left(e^{-4\sigma t} \|Z\|_{H^1 \times L^2}\right) \tag{5.28}$$

$$b_i(t) = \langle Z(t), Z_{-,i}(t) \rangle + O\left(e^{-4\sigma t} \|Z\|_{H^1 \times L^2}\right). \tag{5.29}$$

Proof. This lemma follows from the consideration of the system with unknown variables a_i and b_i which is obtained by replacing E by its definition in (5.26) and (5.27). See also [16] for similar considerations in the case of modulation for the nonlinear Schrödinger equations. ◻

Control of the $Z_{+,i}$ and $Z_{-,i}$ directions

Define $\alpha_{\pm,i} := \langle Z, Z_{\pm,i}\rangle$ for all $i = 1, \dots, N$. We claim:

Lemma 5.6. *The following bounds hold: for all $i = 1, \dots, N$, for all $t \geq t_0$,*

$$|\alpha'_{\pm,i}(t) \mp e_i\alpha_{\pm,i}(t)| \leq C\left(e^{-4\sigma t}\|Z\|_{H^1\times L^2} + \|Z\|^2_{H^1\times L^2}\right).$$

Proof. The proof is in a similar fashion as that of Claim 5.16. We note that

$$\partial_t Z = \begin{pmatrix} 0 & Id \\ \partial_x^2 - Id + f'(\varphi) & 0 \end{pmatrix} Z + \begin{pmatrix} 0 \\ f(u) - f(\varphi) - (u-\varphi)f'(\varphi) \end{pmatrix}$$

and that $|f(u) - f(\varphi) - (u-\varphi)f'(\varphi)| \leq C|u-\varphi|^2 \leq C\|Z\|^2_{H^1\times L^2}$. Thus for $i = 1, \dots, N$, we have

$$\begin{aligned}
\alpha'_{\pm,i} &= \langle \partial_t Z, Z_{\pm,i}\rangle + \langle Z, \partial_t Z_{\pm,i}\rangle \\
&= \left\langle \begin{pmatrix} 0 & Id \\ \partial_x^2 - Id + f'(\varphi) & 0 \end{pmatrix} Z, Z_{\pm,i}\right\rangle - \beta_i\langle Z, \partial_x Z_{\pm,i}\rangle + \mathrm{O}\left(\|Z\|^2_{H^1\times L^2}\right) \\
&= \left\langle Z, \begin{pmatrix} -\beta_i\partial_x & \partial_x^2 - Id + f'(Q_i) \\ Id & -\beta_i\partial_x \end{pmatrix} Z_{\pm,i}\right\rangle + \left\langle Z, \begin{pmatrix} 0 & f'(\varphi) - f'(Q_i) \\ 0 & 0 \end{pmatrix} Z_{\pm,i}\right\rangle \\
&\quad + \mathrm{O}\left(\|Z\|^2_{H^1\times L^2}\right) \\
&= \langle Z, \pm e_i Z_{\pm,i}\rangle + \mathrm{O}\left(e^{-4\sigma t}\|Z\|_{H^1\times L^2} + \|Z\|^2_{H^1\times L^2}\right).
\end{aligned}$$

□

Control of the remaining modulation parameters

Lemma 5.7. *For all $i = 1, \dots, N$, we have*

$$|a'_i| \leq C\left(\|E\|_{H^1\times L^2} + \|Z\|^2_{H^1\times L^2}\right). \tag{5.30}$$

Proof. We do not detail the proof of this lemma which is similar to Claim 5.21. It suffices to start by differentiating the orthogonality relation $\langle E, (R_i)_x\rangle = 0$ with respect to t and then to control terms by means of $\|E\|_{H^1\times L^2}$. □

Study of a Lyapunov functional

Taking some inspiration in [76, 110, 111], we consider for all $t \geq t_0$:

$$\mathcal{F}_z(t) := \int_{\mathbb{R}} \left\{\partial_x z^2 + \partial_t z^2 + z^2 - f'(\varphi)z^2\right\}\, dx + 2\int_{\mathbb{R}} \chi\partial_x z\partial_t z\, dx,$$

where χ is defined as follows.

To begin with, recall that the parameters are ordered in such a way: $-1 < \beta_{\eta(1)} < \cdots < \beta_{\eta(N)} < 1$; let us denote, for some small $\delta > 0$ which will be determined later:

$$\bar{l}_i := \beta_{\eta(i)} + \delta\left(\beta_{\eta(i+1)} - \beta_{\eta(i)}\right)$$

$$\underline{l}_i := \beta_{\eta(i)} - \delta\left(\beta_{\eta(i+1)} - \beta_{\eta(i)}\right).$$

We then define for all $t \geq t_0$ and for all $x \in \mathbb{R}$:

$$\chi(t,x) := \begin{cases} \beta_{\eta(1)} & \text{if} \quad x \in (-\infty, \bar{l}_1 t] \\ \beta_{\eta(i)} & \text{if} \quad x \in [\underline{l}_i t, \bar{l}_i t] \\ \beta_{\eta(N)} & \text{if} \quad x \in [\bar{l}_N t, +\infty) \\ \dfrac{x}{(1-2\delta)t} - \dfrac{\delta}{1-2\delta}\left(\beta_{\eta(i)} + \beta_{\eta(i+1)}\right) & \text{if} \quad x \in [\bar{l}_i t, \underline{l}_{i+1} t], i \in \{1, \ldots, N-1\}. \end{cases}$$

For all $t \geq t_0$, $\chi(t)$ is a piecewise $\mathscr{C}^1$ function.

Set $\Omega(t) := \bigcup_{i=1}^N (\bar{l}_i t, \underline{l}_{i+1} t)$. It follows from the definition of χ that

$$\partial_t \chi(t,x) = \partial_x \chi(t,x) = 0 \quad \text{if } x \in \Omega(t)^{\complement}$$
$$\partial_x \chi(t,x) = \frac{1}{(1-2\delta)t}, \quad \partial_t \chi(t,x) = -\frac{x}{(1-2\delta)t^2} \quad \text{if } x \in \Omega(t).$$

Lemma 5.8. *There exists $\gamma > 0$ such that*

$$\mathcal{F}_z'(t) = 2\int_{\Omega(t)} \partial_x z \partial_t z \partial_t \chi \; dx - \int_{\Omega(t)} \left\{(\partial_t z)^2 + (\partial_x z)^2 - z^2 + f'(\varphi) z^2\right\} \partial_x \chi \; dx$$
$$+ \mathrm{O}\left(e^{-\gamma t}\|Z\|_{H^1\times L^2}^2 + \|Z\|_{H^1\times L^2}^3\right). \quad (5.31)$$

Proof. We essentially have to use the identity $\partial_t^2 z = \partial_x^2 z - z + f(u) - f(\varphi)$ in the expression of $\mathcal{F}_z'(t)$.

We compute

$$\begin{aligned} \mathcal{F}_z'(t) &= 2\int_{\mathbb{R}} \left\{ z_{tx} z_x + z_{tt} z_t + z_t z - f'(\varphi) z_t z - \frac{1}{2}\varphi_t f''(\varphi) z^2 \right\} \; dx \\ &\quad + 2\int_{\mathbb{R}} \chi_t z_x z_t \; dx + 2\int_{\mathbb{R}} \chi z_x z_{tt} \; dx + 2\int_{\mathbb{R}} \chi z_{xt} z_t \; dx \\ &= 2\int_{\mathbb{R}} z_t \left(-z_{xx} + z_{tt} + z - f'(\varphi) z\right) \; dx + 2\int_{\mathbb{R}} \chi_t z_x z_t \; dx \\ &\quad + 2\int_{\mathbb{R}} \chi z_x \left(z_{xx} - z + f(u) - f(\varphi)\right) \; dx - \int_{\mathbb{R}} z_t^2 \chi_x \; dx - \int_{\mathbb{R}} \varphi_t f''(\varphi) z^2 \; dx \end{aligned} \quad (5.32)$$

Notice that

$$\int_{\mathbb{R}} z_t \left(-z_{xx} + z_{tt} + z - f'(\varphi) z\right) \; dx = \int_{\mathbb{R}} z_t \left(f(u) - f(\varphi) - f'(\varphi) z\right) \; dx = \mathrm{O}\left(\|Z\|_{H^1\times L^2}^3\right) \quad (5.33)$$

and

$$\int_{\mathbb{R}} \chi z_x \left(f(u) - f(\varphi)\right) \; dx = \int_{\mathbb{R}} \chi z_x f'(\varphi) z \; dx + \mathrm{O}\left(\|Z\|_{H^1\times L^2}^3\right). \quad (5.34)$$

Hence, collecting (5.32), (5.33), and (5.34),

$$
\begin{aligned}
\mathcal{F}_z'(t) &= 2\int_{\mathbb{R}} \chi_t z_x z_t \, dx - \int_{\mathbb{R}} \chi_x (z_x^2 + z_t^2 - z^2)\, dx \\
&\quad - \int_{\mathbb{R}} z^2 \left(\chi_x f'(\varphi) + \chi \varphi_x f''(\varphi)\right) \, dx - \int_{\mathbb{R}} \varphi_t f''(\varphi) z^2 \, dx + \mathrm{O}\left(\|Z\|_{H^1\times L^2}^3\right) \\
&= 2\int_{\mathbb{R}} z_x z_t \chi_t \, dx - \int_{\mathbb{R}} \left\{z_x^2 + z_t^2 - z^2 + f'(\varphi) z^2\right\} \chi_x \, dx \\
&\quad - \int_{\mathbb{R}} z^2 f''(\varphi)(\varphi_t + \chi\varphi_x)\, dx + \mathrm{O}\left(\|Z\|_{H^1\times L^2}^3\right).
\end{aligned}
\tag{5.35}
$$

Lastly, observe that

$$
\begin{aligned}
\int_{\mathbb{R}} z^2 f''(\varphi)(\varphi_t + \chi\varphi_x)\, dx &= \sum_{i=1}^N \int_{\mathbb{R}} z^2 f''(\varphi) \left((R_k)_t + \chi (R_k)_x\right) \, dx + \mathrm{O}\left(\|Z\|_{H^1\times L^2}^3\right) \\
&= I + J + \mathrm{O}\left(\|Z\|_{H^1\times L^2}^3\right),
\end{aligned}
$$

where

$$
\begin{cases}
I = \displaystyle\sum_{k=1}^N \int_{\Omega(t)} z^2 f''(\varphi) \left((R_k)_t + \chi (R_k)_x\right) \, dx \\
J = \displaystyle\sum_{k=1}^N \sum_{i=1}^N \int_{\underline{l}_i(t)}^{\bar{l}_i t} z^2 f''(\varphi) \left((R_k)_t + \chi (R_k)_x\right) \, dx.
\end{cases}
$$

On the one hand,

$$
\begin{aligned}
J &= \sum_{i=1}^N \sum_{k=1}^N \int_{\underline{l}_i(t)}^{\bar{l}_i t} z^2 f''(\varphi) \left((R_k)_t + \chi (R_k)_x\right) \, dx \\
&= \sum_{i=1}^N \int_{\underline{l}_i(t)}^{\bar{l}_i t} z^2 f''(\varphi) \left((R_{\eta(i)})_t + \chi (R_{\eta(i)})_x\right) \, dx + \mathrm{O}\left(e^{-4\sigma t}\|Z\|_{H^1\times L^2}^2\right) \\
&= \mathrm{O}\left(e^{-4\sigma t}\|Z\|_{H^1\times L^2}^2\right).
\end{aligned}
\tag{5.36}
$$

Indeed, for all $x \in [\underline{l}_i(t), \bar{l}_i t]$, we have $(R_{\eta(i)})_t(t,x) + \chi(t,x)(R_{\eta(i)})_x(t,x) = 0$.

On the other, for $x \in \Omega(t)$, there exists $k' \in \{1, \dots, N\}$ such that $\bar{l}_{k'} t \le x \le \underline{l}_{k'+1} t$. Then

$$
(\bar{l}_{k'} - \beta_k) t \le x - \beta_k t \le (\underline{l}_{k'+1} - \beta_k) t
$$

and thus

$$
|x - \beta_k t| \ge \delta \min_{j=1,\dots,N-1} \{\beta_{\eta(j+1)} - \beta_{\eta(j)}\} t > 0.
$$

As a consequence, we have for all $x \in \Omega(t)$

$$
\begin{aligned}
|(R_k)_x(t,x)| &\le C e^{-\sigma|x-\beta_k t|} \\
&\le C e^{-\gamma t},
\end{aligned}
\tag{5.37}
$$

where $0 < \gamma < \sigma\delta \min_{j=1,\dots,N-1}\{\beta_{\eta(j+1)} - \beta_{\eta(j)}\}$.

Hence

$$|I| \leq Ce^{-\gamma t}\|Z\|^2_{H^1\times L^2}. \tag{5.38}$$

Finally we gather (5.35), (5.36), and (5.38) in order to obtain (5.31). □

Let us introduce the components ε and ε_2 of the vector

$$E = \begin{pmatrix} \varepsilon := z - \sum_i \left\{a_i \partial_x Q_i + b_i (Y_{+,i})_1\right\} \\ \varepsilon_2 := \partial_t z - \sum_i \left\{a_i(-\beta_i \partial_{xx} Q_i) + b_i (Y_{+,i})_2\right\} \end{pmatrix}.$$

Corollary 5.9. *We have*

$$\mathcal{F}_z'(t) = 2\int_{\Omega(t)} \varepsilon_x \varepsilon_2 \chi_t \; dx - \int_{\Omega(t)} \left\{\varepsilon_2^2 + \varepsilon_x^2 - \varepsilon^2 + f'(\varphi)\varepsilon^2\right\}\chi_x \; dx \\ + \mathrm{O}\left(e^{-\gamma t}\|Z\|^2_{H^1\times L^2} + \|Z\|^3_{H^1\times L^2}\right). \tag{5.39}$$

Proof. The corollary immediately follows from (5.31) and bounds for the derivatives of R_k and $Y_{+,k}$ which are analogous to (5.37). □

In the spirit of [76, Proposition 4.2], we will state an almost monotonicity property satisfied by $\mathcal{F}_z$. Let us define

$$\mathcal{F}_{\varepsilon,\Omega(t)}(t) := \int_{\Omega(t)} \left\{\varepsilon_x^2 + \varepsilon_2^2 + 2\chi\varepsilon_x\varepsilon_2\right\} \; dx.$$

Let $\lambda \in (1, \alpha - 1)$. The choice of λ is linked with the integrability of particular quantities and will appear naturally later.

Proposition 5.23. *There exists $\delta > 0$ and $t_0 > 0$ such that for all $t \geq t_0$,*

$$-\mathcal{F}_z'(t) \leq \frac{\lambda}{t}\mathcal{F}_{\varepsilon,\Omega(t)}(t) + \mathrm{O}\left(e^{-\gamma t}\|Z\|^2_{H^1\times L^2} + \|Z\|^3_{H^1\times L^2}\right). \tag{5.40}$$

Proof. Since $f'(0) = 0$ and f' is continuous, there exists $r_0 > 0$ such that for all $r \in [0, r_0]$, $|f'(r)| \leq 1$. There exists $K > 0$ (independent of t) such that for all $t \geq t_0$ and for all $x \in \Omega(t)$, $|\varphi(t,x)| \leq Ke^{-\gamma t}$. Even if it means increasing t_0, we can assume that $Ke^{-\gamma t} \leq r_0$ for all $t \geq t_0$.

In addition $\chi_x \geq 0$. Thus, for $t \geq t_0$,

$$-2\int_{\Omega(t)} \varepsilon_x\varepsilon_2\chi_t \; dx + \int_{\Omega(t)} \left\{\varepsilon_x^2 + \varepsilon_2^2 - \varepsilon^2 + f'(\varphi)\varepsilon^2\right\}\chi_x \; dx \\ \leq -2\int_{\Omega(t)} \varepsilon_x\varepsilon_2\chi_t \; dx + \int_{\Omega(t)} \left\{\varepsilon_x^2 + \varepsilon_2^2\right\}\chi_x \; dx.$$

Moreover,

$$-2\int_{\Omega(t)} \varepsilon_x\varepsilon_2\chi_t \; dx + \int_{\Omega(t)} \left\{\varepsilon_x^2 + \varepsilon_2^2\right\}\chi_x \; dx = \frac{1}{(1-2\delta)t}\int_{\Omega(t)} \left\{2\varepsilon_x\varepsilon_2\frac{x}{t} + \varepsilon_x^2 + \varepsilon_2^2\right\} \; dx \\ = \frac{1}{(1-2\delta)t}\left(\mathcal{F}_{\varepsilon,\Omega(t)} + 2\int_{\Omega(t)} \left(\frac{x}{t} - \chi\right)\varepsilon_x\varepsilon_2 \; dx\right).$$

Now, for $x \in \Omega(t)$, we have

$$\begin{aligned}\left|\frac{x}{t} - \chi(t,x)\right| &\leq \left|\frac{x}{t}\left(1 - \frac{1}{1-2\delta}\right)\right| + \frac{\delta}{1-2\delta} \times 2 \max_i |\beta_i| \\ &\leq 2\delta\left(\left|\frac{x}{t}\right| + C\right) \\ &\leq C\delta.\end{aligned}$$

Thus,

$$\frac{2}{(1-2\delta)t}\int_{\Omega(t)}\left(\frac{x}{t} - \chi\right)\varepsilon_x\varepsilon_2\,dx = \mathrm{O}\left(\delta\|E\|^2_{H^1\times L^2}\right).$$

Noticing moreover that

$$\begin{aligned}\mathcal{F}_{\varepsilon,\Omega(t)} &\geq \int_{\Omega(t)}\left\{\varepsilon_x^2 + \varepsilon_2^2\right\}\,dx - 2\|\chi(t)\|_\infty\int_{\Omega(t)}\varepsilon_x\varepsilon_2\,dx \\ &\geq \int_{\Omega(t)}\left\{\varepsilon_x^2 + \varepsilon_2^2\right\}\,dx - \|\chi(t)\|_\infty\int_{\Omega(t)}\left\{\varepsilon_x^2 + \varepsilon_2^2\right\}\,dx \\ &\geq (1 - \|\chi(t)\|_\infty)\,\|E\|^2_{H^1\times L^2(\Omega(t))},\end{aligned}$$

we obtain

$$\frac{2}{(1-2\delta)t}\int_{\Omega(t)}\left(\frac{x}{t} - \chi\right)\varepsilon_x\varepsilon_2\,dx = \mathrm{O}\left(\delta\mathcal{F}_{\varepsilon,\Omega(t)}\right).$$

Finally,

$$\begin{aligned}-\mathcal{F}_z'(t) &\leq \frac{1}{(1-2\delta)t}(1+\mathrm{O}(\delta))\mathcal{F}_{\varepsilon,\Omega(t)} + \mathrm{O}\left(e^{-\gamma t}\|Z\|^2_{H^1\times L^2} + \|Z\|^3_{H^1\times L^2}\right) \\ &\leq \frac{\lambda}{t}\mathcal{F}_{\varepsilon,\Omega(t)} + \mathrm{O}\left(e^{-\gamma t}\|Z\|^2_{H^1\times L^2} + \|Z\|^3_{H^1\times L^2}\right),\end{aligned}$$

provided δ is chosen small enough. □

We now introduce the functional

$$\mathcal{F}_\varepsilon(t) := \int_{\mathbb{R}}\left\{\varepsilon_x^2 + \varepsilon_2^2 + \varepsilon^2 - f'(\varphi)\varepsilon^2\right\}\,dx + 2\int_{\mathbb{R}}\chi\varepsilon_x\varepsilon_2\,dx$$

and compare it with $\mathcal{F}_z$.

Lemma 5.10. *We have*

$$\mathcal{F}_\varepsilon(t) = \mathcal{F}_z(t) - 2\sum_{i=1}^N \alpha_{-,i}(t)\alpha_{+,i}(t) + \mathcal{G}(t),$$

where $\mathcal{G}(t) = \mathrm{O}\left(e^{-\gamma t}\|Z\|^2_{H^1\times L^2}\right)$ *and* $\mathcal{G}'(t) = \mathrm{O}\left(e^{-\gamma t}\|Z\|^2_{H^1\times L^2}\right)$.

Proof. Let M be the matrix $\begin{pmatrix} -\partial_x^2 + Id - f'(\varphi) & 0 \\ 0 & Id \end{pmatrix}$. For all $i = 1, \ldots, N$, we have the decomposition

$$M = H_i + \begin{pmatrix} 0 & \beta_i\partial_x \\ -\beta_i\partial_x & 0 \end{pmatrix} + \begin{pmatrix} f'(Q_i) - f'(\varphi) & 0 \\ 0 & 0 \end{pmatrix}. \tag{5.41}$$

Then we infer

$$\begin{aligned}
\mathcal{F}_\varepsilon(t) &= \langle ME, E\rangle + 2\int_{\mathbb{R}} \chi \varepsilon_x \varepsilon_2 \, dx \\
&= \langle MZ, Z\rangle - 2\left\langle M\left(\sum_{i=1}^N \left\{a_i (R_i)_x + b_i Y_{+,i}\right\}\right), Z\right\rangle \\
&\quad + \left\langle M\left(\sum_{i=1}^N \left\{a_i (R_i)_x + b_i Y_{+,i}\right\}\right), \sum_{j=1}^N \left\{a_j (R_j)_x + b_j Y_{+,j}\right\}\right\rangle \\
&\quad + 2\int_{\mathbb{R}} \chi z_x z_t \, dx + 2\int_{\mathbb{R}} \chi \sum_{i=1}^N \left\{a_i (Q_i)_{xx} + b_i (Y_{+,i})_{1,x}\right\} \sum_{j=1}^N \left\{a_j (Q_j)_{xt} + b_j (Y_{+,j})_2\right\} \, dx \\
&\quad - 2\int_{\mathbb{R}} \chi z_x \sum_{i=1}^N \left\{a_i (Q_i)_{xt} + b_i (Y_{+,i})_2\right\} \, dx - 2\int_{\mathbb{R}} \chi z_t \sum_{i=1}^N \left\{a_i (Q_i)_{xx} + b_i (Y_{+,i})_{1,x}\right\} \, dx \\
&= \mathcal{F}_z(t) - 2\sum_{i=1}^N b_i \langle Z_{+,i}, Z\rangle + \tilde{\mathcal{G}}(t) \\
&= \mathcal{F}_z(t) - 2\sum_{i=1}^N \alpha_{-,i}(t)\alpha_{+,i}(t) + \mathcal{G}(t),
\end{aligned}$$

with

$$\begin{aligned}
MY_{+,i} &= Z_{+,i} + \beta_i \begin{pmatrix} (Y_{+,i})_{2,x} \\ -(Y_{+,i})_{1,x} \end{pmatrix} + \begin{pmatrix} (f'(Q_i) - f'(\varphi))(Y_{+,i})_1 \\ 0 \end{pmatrix} \\
M(R_i)_x &= \begin{pmatrix} -\beta_i^2 (Q_i)_{xxx} \\ -\beta_i (Q_i)_{xx} \end{pmatrix} + \begin{pmatrix} (f'(Q_i) - f'(\varphi))(Q_i)_x \\ 0 \end{pmatrix}.
\end{aligned}$$

and

$$\begin{aligned}
\tilde{\mathcal{G}}(t) &= -2\left\langle \sum_{i=1}^N \left\{ a_i \begin{pmatrix} -\beta_i^2 (Q_i)_{xxx} \\ \beta_i (Q_i)_{xx} \end{pmatrix} + b_i\beta_i \begin{pmatrix} (Y_{+,i})_{2,x} \\ -(Y_{+,i})_{1,x} \end{pmatrix} \right\}, Z \right\rangle + \mathrm{O}\left(e^{-\gamma t} \|Z\|^2_{H^1\times L^2}\right) \\
&\quad + \sum_{i=1}^N \left\langle a_i \begin{pmatrix} -\beta_i^2 (Q_i)_{xxx} \\ \beta_i (Q_i)_{xx} \end{pmatrix} + b_i\beta_i \begin{pmatrix} (Y_{+,i})_{2,x} \\ -(Y_{+,i})_{1,x} \end{pmatrix} + b_i Z_{+,i}, a_i (R_i)_x + b_i Y_{+,i} \right\rangle \\
&\quad + 2\sum_{i=1}^N \int_{\mathbb{R}} \chi \left(a_i (Q_i)_{xx} + b_i (Y_{+,i})_{1,x}\right)\left(-\beta_i a_i (Q_i)_{xx} + b_i (Y_{+,i})_2\right) \, dx \\
&\quad - 2\sum_{i=1}^N \int_{\mathbb{R}} \chi z_x \left(-\beta_i a_i (Q_i)_{xx} + b_i (Y_{+,i})_2\right) \, dx - 2\sum_{i=1}^N \int_{\mathbb{R}} \chi z_t \left(a_i (Q_i)_{xx} + b_i (Y_{+,i})_{1,x}\right) \, dx.
\end{aligned}$$

Integrating by parts, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}} \chi z_x \left(-\beta_i a_i (Q_i)_{xx} + b_i (Y_{+,i})_2\right) \, dx \\
&\qquad = -\int_{\mathbb{R}} \chi z \left(-\beta_i a_i (Q_i)_{xxx} + b_i (Y_{+,i})_{2,x}\right) \, dx - \int_{\mathbb{R}} \chi_x z \left(-\beta_i a_i (Q_i)_{xx} + b_i (Y_{+,i})_2\right) \, dx.
\end{aligned}$$

Consequently,

$$
\begin{aligned}
\tilde{\mathcal{G}}(t) =& -2\sum_{i=1}^{N} a_i \int_{\mathbb{R}} (\chi - \beta_i)\left(-\beta_i z (Q_i)_{xxx} + z_t (Q_i)_{xx}\right) dx \\
& - 2\sum_{i=1}^{N} b_i \int_{\mathbb{R}} (\chi - \beta_i)\left(-z (Y_{+,i})_{2,x} + z_t (Y_{+,i})_{1,x}\right) dx \\
& + 2\sum_{i=1}^{N} a_i^2 \beta_i \int_{\mathbb{R}} (\beta_i - \chi)(Q_i)_{xx}^2 \, dx + 2\sum_{i=1}^{N} b_i^2 \int_{\mathbb{R}} (\beta_i - \chi)(Y_{+,i})_1 (Y_{+,i})_{2,x} \\
& + 2\sum_{i=1}^{N} \int_{\Omega(t)} \chi_x z \left(-\beta_i a_i (Q_i)_{xx} + b_i (Y_{+,i})_2\right) dx \\
& + 2\sum_{i=1}^{N} a_i b_i \int_{\mathbb{R}} (\chi - \beta_i)\left(-\beta_i (Y_{+,i})_{1,x} (Q_i)_{xx} + (Q_i)_{xx} (Y_{+,i})_2\right) dx + \mathrm{O}\left(e^{-\gamma t} \|Z\|_{H^1 \times L^2}^2\right) \\
=& \, \mathrm{O}\left(e^{-\gamma t} \|Z\|_{H^1 \times L^2}^2\right).
\end{aligned}
$$

□

We deduce from Proposition 5.23 and Lemma 5.10 the following "weak" monotonicity property.

Corollary 5.11. *We have for all $t \geq t_0$,*

$$
-\mathcal{F}_\varepsilon'(t) \leq \frac{\lambda}{t}\mathcal{F}_\varepsilon(t) + \frac{C}{t}\sum_{i=1}^{N} \alpha_{+,i}^2 + \mathrm{O}\left(e^{-\gamma t}\|Z\|_{H^1\times L^2}^2 + \|Z\|_{H^1\times L^2}^3\right). \tag{5.42}
$$

Proof. From Lemma 5.6, we obtain: for all $i = 1, \dots, N$, for all $t \geq t_0$,

$$
\left|(\alpha_{-,i}\alpha_{+,i})'\right| \leq C\left(e^{-\gamma t}\|Z\|_{H^1\times L^2}^2 + \|Z\|_{H^1\times L^2}^3\right). \tag{5.43}
$$

Thus, we have

$$
\begin{aligned}
-\mathcal{F}_\varepsilon'(t) &= -\mathcal{F}_z'(t) + 2\sum_{i=1}^{N} (\alpha_{-,i}\alpha_{+,i})' + \mathrm{O}\left(e^{-\gamma t}\|Z\|^2\right) \\
&\leq \frac{\lambda}{t}\mathcal{F}_{\varepsilon,\Omega(t)} + \mathrm{O}\left(e^{-\gamma t}\|Z\|_{H^1\times L^2}^2 + \|Z\|_{H^1\times L^2}^3\right).
\end{aligned}
$$

We now make use of the following property satisfied by $\mathcal{F}_\varepsilon$ which is a consequence of a localized version of Proposition 5.9 (we refer to [76, proof of (4.12) and (4.21)] for similar considerations in the case of the energy-critical wave equation):

$$
\mathcal{F}_{\varepsilon,\Omega(t)} \leq \mathcal{F}_\varepsilon(t) + C\sum_{i=1}^{N} \langle \varepsilon, Z_{+,i}\rangle^2, \tag{5.44}
$$

to deduce that

$$
-\mathcal{F}_\varepsilon'(t) \leq \frac{\lambda}{t}\mathcal{F}_\varepsilon(t) + \frac{C}{t}\sum_{i=1}^{N} \alpha_{+,i}^2 + \mathrm{O}\left(e^{-\gamma t}\|Z\|_{H^1\times L^2}^2 + \|Z\|_{H^1\times L^2}^3\right).
$$

□

We are now in a position to prove

Proposition 5.24. *Even if it means taking a larger t_0, we have for all $t \geq t_0$*

$$\|Z(t)\|_{H^1\times L^2} \leq C \sup_{t'\geq t} \sum_{i=1}^{N} |\alpha_{-,i}(t')|. \tag{5.45}$$

Proof. Multiplying the estimate obtained in Corollary 5.11 by t^λ, we have for all $t \geq t_0$,

$$-(t^\lambda \mathcal{F}_\varepsilon)'(t) \leq C \left(\sum_{i=1}^{N} t^{\lambda-1}\alpha_{+,i}^2 + t^\lambda e^{-\gamma t}\|Z\|^2_{H^1\times L^2} + t^\lambda \|Z\|^3_{H^1\times L^2} \right).$$

Since, $t^{\lambda-1}\alpha_{+,i}^2$, $t^\lambda\|Z\|^3_{H^1\times L^2}$ are integrable functions of t on $[t_0,+\infty)$ and $t^\lambda\mathcal{F}_\varepsilon(t) \to 0$ as $t \to +\infty$, we infer that

$$\mathcal{F}_\varepsilon(t) \leq C \left(\frac{1}{t^\lambda} \sum_{i=1}^{N} \int_t^{+\infty} t'^{\lambda-1}\alpha_{+,i}^2(t')\,dt' + \frac{1}{t^\lambda}\int_t^{+\infty} t'^\lambda \|Z(t')\|^3_{H^1\times L^2}\,dt' + e^{-\gamma t}\sup_{t'\geq t}\|Z(t')\|^2_{H^1\times L^2} \right). \tag{5.46}$$

By the coercivity property satisfied by $\mathcal{F}_\varepsilon$, we thus obtain

$$\begin{aligned}
\|E(t)\|^2_{H^1\times L^2} &\leq C\left(\mathcal{F}_\varepsilon(t) + \sum_{i=1}^{N}\alpha_{+,i}^2(t) + e^{-2\gamma t}\|Z(t)\|^2_{H^1\times L^2} \right) \\
&\leq \frac{C}{t^\lambda}\sum_{i=1}^{N}\int_t^{+\infty} t'^{\lambda-1}\alpha_{+,i}^2(t')\,dt' + \frac{C}{t^\lambda}\int_t^{+\infty} t'^\lambda\|Z(t')\|^3_{H^1\times L^2}\,dt' \\
&\quad + Ce^{-\gamma t}\sup_{t'\geq t}\|Z(t')\|^2_{H^1\times L^2} + C\sum_{i=1}^{N}\alpha_{+,i}^2(t).
\end{aligned}$$

In other words, there exists $C \geq 0$ such that for all $t \geq t_0$,

$$\begin{aligned}
\|E(t)\|_{H^1\times L^2} \leq &\frac{C}{t^{\frac{\lambda}{2}}}\left(\int_t^{+\infty} t'^{\lambda-1}\sum_{i=1}^{N}\alpha_{+,i}^2(t')\,dt' \right)^{\frac12} + C\left(\sum_{i=1}^{N}\alpha_{+,i}^2(t) \right)^{\frac12} \\
&+ \frac{C}{t^{\frac{\lambda}{2}}}\left(\int_t^{+\infty} t'^\lambda \|Z(t')\|^3_{H^1\times L^2}\,dt' \right)^{\frac12} + Ce^{-\frac{\gamma}{2}t}\sup_{t'\geq t}\|Z(t')\|_{H^1\times L^2}.
\end{aligned} \tag{5.47}$$

From Lemma 5.6 we recall the estimate

$$|\alpha'_{+,i}(t) - e_i\alpha_{+,i}(t)| \leq C\left(e^{-\gamma t}\|Z(t)\|_{H^1\times L^2} + \|Z(t)\|^2_{H^1\times L^2} \right)$$

which is equivalent to

$$|(e^{-e_i t}\alpha_{+,i})'(t)| \leq Ce^{-e_i t}\left(e^{-\gamma t}\|Z(t)\|_{H^1\times L^2} + \|Z(t)\|^2_{H^1\times L^2} \right).$$

Integrating the preceding inequality (which is indeed possible), we deduce:

$$|\alpha_{+,i}(t)| \leq C\left(e^{-\gamma t}\sup_{t'\geq t}\|Z(t')\|_{H^1\times L^2} + \sup_{t'\geq t}\|Z(t')\|^2_{H^1\times L^2} \right). \tag{5.48}$$

Now,

$$\|Z(t)\|_{H^1\times L^2} \leq \|E(t)\|_{H^1\times L^2} + C\sum_{i,\pm}|\alpha_{\pm,i}(t)| + C\sum_{i=1}^{N}|a_i(t)| + Ce^{-\gamma t}\|Z(t)\|_{H^1\times L^2}.$$

By Lemma 5.7, it follows that

$$\|Z(t)\|_{H^1\times L^2} \leq C\left(\|E(t)\|_{H^1\times L^2} + \sum_{\pm,i}|\alpha_{\pm,i}(t)| + \int_t^{+\infty}\left(\|E(t')\|_{H^1\times L^2} + \|Z(t')\|^2_{H^1\times L^2}\right)\,dt'\right). \tag{5.49}$$

Observing that the quantity $\int_t^{+\infty}\|Z(t')\|_{H^1\times L^2}\,dt'$ makes sense and tends to 0 as $t\longrightarrow+\infty$ (because $\alpha>1$) and that

$$\int_t^{+\infty}\|Z(t')\|^2_{H^1\times L^2}\,dt' \leq \sup_{t'\geq t}\|Z(t')\|_{H^1\times L^2}\int_t^{+\infty}\|Z(t')\|_{H^1\times L^2}\,dt,$$

we deduce that for t sufficiently large

$$\|Z(t)\|_{H^1\times L^2} \leq C\left(\sup_{t'\geq t}\|E(t')\|_{H^1\times L^2} + \int_t^{+\infty}\|E(t')\|_{H^1\times L^2}\,dt' + \sup_{t'\geq t}\sum_{\pm,i}|\alpha_{\pm,i}(t')|\right). \tag{5.50}$$

Now, in order to obtain an estimate of $\|E(t)\|_{H^1\times L^2}$ and of $\int_t^{+\infty}\|E(t')\|_{H^1\times L^2}\,dt'$ in terms of $\|Z(t)\|_{H^1\times L^2}$ and $\sup_{t'\geq t}\|Z(t')\|_{H^1\times L^2}$, we replace (5.48) in (5.47). We notice that the following well-defined quantities tend to 0 as $t\longrightarrow+\infty$ (because $\alpha>3$ and by the choice of $\lambda<\alpha-1$):

$$\int_t^{+\infty} t'^{\lambda-1}\sup_{t''\geq t'}\|Z(t'')\|^2_{H^1\times L^2}\,dt', \qquad \int_t^{+\infty}\sup_{t''\geq t'}\|Z(t'')\|_{H^1\times L^2}\,dt'$$

$$\int_t^{+\infty} t'^{\lambda}\|Z(t')\|_{H^1\times L^2}\,dt', \qquad \int_t^{+\infty}\frac{1}{u^{\frac{\lambda}{2}}}\left(\int_u^{+\infty} t'^{\lambda-1}\sup_{t''\geq t'}\|Z(t'')\|^2_{H^1\times L^2}\,dt'\right)^{\frac12}\,du$$

$$\int_t^{+\infty}\frac{1}{u^{\frac{\lambda}{2}}}\left(\int_u^{+\infty} t'^{\lambda}\|Z(t')\|_{H^1\times L^2}\,dt'\right)^{\frac12}\,du.$$

and use (5.50). We then obtain for t sufficiently large

$$\|Z(t)\|_{H^1\times L^2} \leq C\sup_{t'\geq t}\sum_{i=1}^{N}|\alpha_{-,i}(t')|.$$

□

Then, we claim

Proposition 5.25. *We have for all $t\geq t_0$, for all $i=1,\ldots,N$,*

$$|\alpha_{-,i}(t)| \leq Ce^{-e_1 t}. \tag{5.51}$$

Proof. From Lemma 5.6 and Proposition 5.24, it results that for all $i = 1, \dots, N$, for all $t \geq t_0$,

$$|\alpha'_{-,i}(t) + e_i \alpha_{-,i}(t)| \leq C \left(e^{-\sigma t} \sup_{t' \geq t} \sum_{j=1}^{N} |\alpha_{-,j}(t')| + \left(\sup_{t' \geq t} \sum_{j=1}^{N} |\alpha_{-,j}(t')| \right)^2 \right). \tag{5.52}$$

Then, for all $i = 1, \dots, N$,

$$|\alpha'_{-,i}(t)\alpha_{-,i}(t) + e_i \alpha^2_{-,i}(t)| \leq C \left(e^{-\sigma t} \sup_{t' \geq t} \sum_{j=1}^{N} |\alpha_{-,j}(t')||\alpha_{-,i}(t)| + \sup_{t' \geq t} \left(\sum_{j=1}^{N} |\alpha_{-,j}(t')| \right)^2 |\alpha_{-,i}(t)| \right). \tag{5.53}$$

Let us denote $\mathcal{A} := \sum_{j=1}^{N} \alpha^2_{-,j}$. Summing on $i = 1, \dots, N$, we have in particular:

$$|\mathcal{A}'(t) + 2e_1 \mathcal{A}(t)| \leq C e^{-\sigma t} \left(\sup_{t' \geq t} \sum_{j=1}^{N} |\alpha_{-,j}(t')| \right) \sum_{i=1}^{N} |\alpha_{-,i}(t)| + C \left(\sup_{t' \geq t} \sum_{j=1}^{N} |\alpha_{-,j}(t')| \right)^2 \sum_{i=1}^{N} |\alpha_{-,i}(t)|.$$

Noticing that $\left(\sum_{j=1}^{N} |\alpha_{-,j}| \right)^2 \leq C\mathcal{A}$, we obtain the existence of $c > 0$ such that for all $t \geq t_0$,

$$|\mathcal{A}'(t) + 2e_1 \mathcal{A}(t)| \leq C \left(e^{-\sigma t} + \mathcal{A}(t)^{\frac{1}{2}} \right) \sup_{t' \geq t} \mathcal{A}(t').$$

Lastly we observe that $\xi : t \mapsto e^{-\sigma t} + \mathcal{A}(t)^{\frac{1}{2}}$ is integrable on $[t_0, +\infty)$ since

$$\mathcal{A}(t)^{\frac{1}{2}} = \mathrm{O}\left(\|Z(t)\|^{\frac{1}{2}}_{H^1 \times L^2} \right) = \mathrm{O}\left(\frac{1}{t^{\frac{\alpha}{2}}} \right).$$

By Lemma 5.26 in Appendix, we obtain $\mathcal{A}(t) \leq C e^{-2e_1 t}$ for $t \geq t_0$. Consequently, for all $i = 1, \dots, N$,

$$|\alpha_{-,i}(t)| \leq C e^{-e_1 t}.$$

□

Gathering Propositions 5.24 and 5.25, we deduce

Proposition 5.26. *There exists $C \geq 0$ such that for t sufficiently large,*

$$\|Z(t)\|_{H^1 \times L^2} \leq C e^{-e_1 t}.$$

5.3.2 Identification of the solution

Recall that we have constructed in Section 5.2 a family of multi-solitons $(\phi_{A_1,\dots,A_N})$ such that for all $j = 1, \dots, N$, for all $t \geq t_0$,

$$\left\| \Phi_{A_1,\dots,A_j}(t) - \Phi_{A_1,\dots,A_{j-1}}(t) - A_j e^{-e_j t} Y_{+,j}(t) \right\|_{H^1 \times L^2} \leq C e^{-(e_j + \sigma)t}. \tag{5.54}$$

(We can always assume that $\sigma < \min \left\{ e_1, \min_{j=2,\dots,N} \{ e_j - e_{j-1} \} \right\}$).

Following the strategy of Combet [11], our goal is to establish

Proposition 5.27. *For all $j = 1, \dots, N$, there exist $C \geq 0$, $t_0 \geq 0$, and $A_1, \dots, A_j \in \mathbb{R}$ such that, defining*

$$E_j := U - \Phi_{A_1,\dots,A_j},$$

we have:

$$\|E_j(t)\|_{H^1 \times L^2} \leq Ce^{-e_j t}. \tag{5.55}$$

Moreover, denoting $\alpha_{\pm,j,k} := \langle E_j, Z_{\pm,k}\rangle$ for all $k = 1, \dots, N$, we have

$$\forall\, k \in \{1, \dots, j\}, \qquad e^{e_k t}\alpha_{-,j,k}(t) \to 0, \quad \text{as } t \to +\infty. \tag{5.56}$$

Proof. We proceed by induction on j. First, we focus on the case where $j = 1$.
We have $\|Z(t)\|_{H^1\times L^2} \leq Ce^{-e_1 t}$ by Proposition 5.26. Thus, by Lemma 5.6 and given that $\sigma < e_1$,

$$\left|(e^{e_1 t}\alpha_{-,1})'\right| \leq Ce^{-\sigma t}.$$

Since $t \mapsto e^{-\sigma t}$ is integrable in $+\infty$, there exists $A_1 \in \mathbb{R}$ such that

$$e^{e_1 t}\alpha_{-,1}(t) \to A_1, \qquad \text{as } t \to +\infty.$$

We then define $E_1 := U - \Phi_{A_1}$. We notice that $E_1 = E + (\Phi - \Phi_{A_1})$ so that

$$\begin{aligned}
\|E_1(t)\|_{H^1\times L^2} &\leq \|E(t)\|_{H^1\times L^2} + \|\Phi(t) - \Phi_{A_1}(t)\|_{H^1\times L^2} \\
&\leq Ce^{-e_1 t} + \|\Phi_{A_1}(t) - \Phi(t) - A_1 e^{-e_1 t}Y_{+,1}(t)\|_{H^1\times L^2} + \|A_1 e^{-e_1 t}Y_{+,1}(t)\|_{H^1\times L^2} \\
&\leq Ce^{-e_1 t}.
\end{aligned}$$

Moreover

$$\begin{aligned}
\alpha_{-,1,1} &= \langle E, Z_{-,1}\rangle + \langle \Phi - \Phi_{A_1} - A_1 e^{-e_1 t}Y_{+,1}, Z_{-,1}\rangle + A_1 e^{-e_1 t}\langle Y_{+,1}, Z_{-,1}\rangle \\
&= -\,\alpha_{-,1} + A_1 e^{-e_1 t} + \mathrm{O}\left(e^{-(e_1+\sigma)t}\right) \\
&= \mathrm{o}\left(e^{-e_1 t}\right),
\end{aligned}$$

the last line resulting from the definition of A_1.
Thus Proposition 5.27 is true for $j = 1$.

We now assume that there exist $A_1, \dots, A_{j-1} \in \mathbb{R}$ such that $\|E_{j-1}(t)\|_{H^1\times L^2} \leq Ce^{-e_{j-1}t}$ and for all $k = 1, \dots, j-1$, $e^{e_k t}\alpha_{-,j-1,k}(t) \to 0$ as $t \to +\infty$.
Let us show

Claim 5.28. *We have*

$$\|E_{j-1}(t)\|_{H^1\times L^2} \leq Ce^{-e_j t}.$$

- To prove this claim, we show that, if $\|E_{j-1}(t)\| \leq Ce^{-\sigma_0 t}$ with $e_{j-1} < \sigma_0 < e_j - \sigma$, then

$$\|E_{j-1}(t)\|_{H^1\times L^2} \leq Ce^{-(\sigma+\sigma_0)t}.$$

As

$$\left|\alpha'_{\pm,j-1,k}(t) \mp e_k\alpha_{\pm,j-1,k}(t)\right| \leq C\left(e^{-\sigma t}\|E_{j-1}(t)\|_{H^1\times L^2} + \|E_{j-1}(t)\|^2_{H^1\times L^2}\right)$$

$$\leq Ce^{-(\sigma+\sigma_0)t}$$

(by the same calculations and arguments as those developed in the proof of Lemma 5.6), we have for all $k = 1, \dots, j-1$,

$$\left|(e^{e_k t}\alpha_{-,j-1,k})'\right| \leq Ce^{-(\sigma+\sigma_0-e_k)t}.$$

Since $t \mapsto e^{-(\sigma+\sigma_0-e_k)t}$ is integrable in the neighborhood of $+\infty$ (since $e_k \leq e_{j-1}$), and by assumption, $e^{e_k t}\alpha_{-,j-1,k}(t) \to 0$ as $t \to +\infty$, we have by integration

$$|\alpha_{-,j-1,k}(t)| \leq Ce^{-(\sigma+\sigma_0)t}.$$

For all $k = j, \dots, N$, we have $\sigma + \sigma_0 - e_k \leq \sigma + \sigma_0 - e_j < 0$, thus by integration on $[t_0, t]$, we obtain

$$\left|e^{e_k t}\alpha_{-,j-1,k}(t) - e^{e_k t_0}\alpha_{-,j-1,k}(t_0)\right| \leq Ce^{(e_k-\sigma_0-\sigma)t}.$$

Eventually, we obtain (by a "cut-and-paste" of the argument exposed in subsection 5.3.1)

$$\|E_{j-1}(t)\|_{H^1\times L^2} \leq C \sup_{t'\geq t} \sum_{k=1}^{N} |\alpha_{-,j-1,k}(t')| \leq Ce^{-(\sigma_0+\sigma)t},$$

which is what was expected.

- Now, from the preceding induction, there exists $\tilde{\sigma}_0 \in (e_j - \sigma, e_j)$ such that

$$\|E_{j-1}(t)\|_{H^1\times L^2} \leq Ce^{-\tilde{\sigma}_0 t},$$

from which we deduce

$$\left|(e^{e_k t}\alpha_{-,j-1,k})'\right| \leq Ce^{-(\sigma+\tilde{\sigma}_0-e_k)t}.$$

Now, for $k \in \{1, \dots, j-1\}$, $e_k - \sigma - \tilde{\sigma}_0 \leq e_{j-1} - \sigma - e_j < 0$, we thus have

$$|\alpha_{-,j-1,k}(t)| \leq Ce^{-(\sigma_0+\tilde{\sigma})t} \leq Ce^{-e_j t}.$$

For $k = j$, we have $|(e^{e_j t}\alpha_{-,j-1,j})'| \leq Ce^{(e_j-\tilde{\sigma}_0-\sigma)t}$. Thus, there exists $A_j \in \mathbb{R}$ such that

$$e^{e_j t}\alpha_{-,j-1,j}(t) \to A_j, \quad \text{as } t \to +\infty.$$

For $k \in \{j+1, \dots, N\}$, we have $\sigma + \tilde{\sigma}_0 - e_k < \sigma + e_j - e_k < 0$, thus by integration

$$\begin{aligned}|\alpha_{-,j-1,k}(t)| &\leq Ce^{-e_k t} + Ce^{-(\tilde{\gamma}_0+\gamma)t} \\ &\leq Ce^{-e_j t}.\end{aligned}$$

Hence,

$$\|E_{j-1}(t)\|_{H^1\times L^2} \leq C \sup_{t'\geq t} \sum_{k=1}^{N} |\alpha_{-,j-1,k}(t')| \leq Ce^{-e_j t}.$$

Let us conclude the proof of Proposition 5.27. We define at this stage $E_j := U - \Phi_{A_1,\dots,A_j}$. We immediately have

$$E_j(t) = E_{j-1}(t) + \Phi_{A_1,\dots,A_{j-1}}(t) - \Phi_{A_1,\dots,A_j}(t).$$

Then,

$$\begin{aligned}\|E_j(t)\|_{H^1\times L^2} &\le \|E_{j-1}(t)\|_{H^1\times L^2} + \|\Phi_{A_1,\dots,A_j}(t) - \Phi_{A_1,\dots,A_{j-1}}(t) - A_j e^{-e_j t} Y_{+,j}(t)\|_{H^1\times L^2} \\ &\quad + \|A_j e^{-e_j t} Y_{+,j}(t)\|_{H^1\times L^2} \\ &\le C e^{-e_j t}.\end{aligned}$$

What is more,

$$\begin{aligned}\alpha_{-,j,k}(t) &= \langle E_j(t), Z_{-,k}(t)\rangle \\ &= \alpha_{-,j-1,k}(t) - A_j e^{-e_j t}\langle Y_{+,j}, Z_{-,k}\rangle + \mathrm{O}\left(e^{-(e_j+\sigma)t}\right).\end{aligned}$$

For $k = 1, \dots, j-1$, we have:

$$\begin{aligned}e^{e_k t}|\alpha_{-,j,k}(t)| &\le C e^{e_k t}|\alpha_{-,j-1,k}(t)| + \mathrm{O}\left(e^{-(e_j - e_k+\sigma)t}\right). \\ &\le C e^{(e_k - e_j)t} \underset{t\to+\infty}{\longrightarrow} 0.\end{aligned}$$

For $k = j$,

$$e^{e_j t}\alpha_{-,j,j}(t) = e^{e_j t}\alpha_{-,j-1,j}(t) - A_j + \mathrm{O}(e^{-\sigma t}) \underset{t\to+\infty}{\longrightarrow} 0.$$

This finishes the induction argument. □

Finally we obtain that $U = \Phi_{A_1,\dots,A_N}$ by means of

Corollary 5.12. *For t sufficiently large,* $\|E_N(t)\|_{H^1\times L^2} = 0$.

Proof. As in the preceding proofs, the following bounds hold:

$$\|E_N(t)\|_{H^1\times L^2} \le C \sup_{t'\ge t}\sum_{i=1}^{N}|\alpha_{-,N,i}(t')| \tag{5.57}$$

and

$$|\alpha'_{-,N,i}(t) + e_i\alpha_{-,N,i}(t)| \le C\left(e^{-\sigma t}\|E_N(t)\|_{H^1\times L^2} + \|E_N(t)\|^2_{H^1\times L^2}\right). \tag{5.58}$$

We observe that $t \mapsto e^{e_i t}\left(e^{-\sigma t}\|E_N(t)\|_{H^1\times L^2} + \|E_N(t)\|^2_{H^1\times L^2}\right)$ is integrable on $[t_0, +\infty)$; this is due to the fact that $\|E_N(t)\|_{H^1\times L^2} \le Ce^{-e_N t}$. Since for all $i = 1, \dots, N$, $e^{e_i t}\alpha_{-,N,i}(t) \to 0$ as $t \to +\infty$, we obtain by integration of (5.58) on $[t, +\infty)$:

$$|\alpha_{-,N,i}(t)| \le C\left(e^{-\sigma t}\sup_{t'\ge t}\|E_N(t')\|_{H^1\times L^2} + \sup_{t'\ge t}\|E_N(t')\|^2_{H^1\times L^2}\right).$$

Then, using (5.57), we obtain

$$\sup_{t'\ge t}\|E_N(t')\|_{H^1\times L^2} \le C\left(e^{-\sigma t}\sup_{t'\ge t}\|E_N(t')\|_{H^1\times L^2} + \sup_{t'\ge t}\|E_N(t')\|^2_{H^1\times L^2}\right).$$

This implies that $\|E_N(t)\|_{H^1\times L^2} = 0$ for t sufficiently large. □

5.4 Construction of a one-parameter family of solutions converging to a soliton

The goal of this section is to prove the existence part in Theorem 5.6. Once again we restrict our focus to $d = 1$.

5.4.1 Outline of the construction

Let $A \in \mathbb{R}$.

Let $(S_n)_{n\in\mathbb{N}}$ be an increasing sequence of real numbers which tends to $+\infty$ and, for all $n \in \mathbb{N}$, define u_n as the maximal solution of (NLKG) such that

$$U_n(S_n) = R_\beta(S_n) + Ae^{-e_\beta S_n} Y_{+,\beta}(S_n), \tag{5.59}$$

with obvious notations.

We aim at proving the following key proposition:

Proposition 5.29. *There exist $t_0 \geq 0$ and $C_0 \geq 0$ such that for n large,*

$$\forall\, t \in [t_0, S_n], \qquad \left\| U_n(t) - R_\beta(t) - Ae^{-e_\beta t} Y_{+,\beta}(t) \right\|_{H^1\times L^2} \leq C_0 e^{-2e_\beta t}. \tag{5.60}$$

To this end, we will set up a bootstrap argument and show

Proposition 5.30. *There exist $\alpha_0 > 0$, $C_0 > 0$, and $t_0 \geq 0$ such that for n sufficiently large, if there exists $t_n^* \in [t_0, S_n]$ such that for all $t \in [t_n^*, S_n]$,*

$$\| U_n(t) - R_\beta(t) - Ae^{-e_\beta t} Y_{+,\beta}(t) \|_{H^1\times L^2} \leq \alpha_0, \tag{5.61}$$

then for all $t \in [t_n^*, S_n]$,

$$\| U_n(t) - R_\beta(t) - Ae^{-e_\beta t} Y_{+,\beta}(t) \|_{H^1\times L^2} \leq C_0 e^{-2e_\beta t}. \tag{5.62}$$

Let us show how to deduce Proposition 5.29 from Proposition 5.30.

Proof of Proposition 5.29. Assume momentarily that Proposition 5.30 holds true. Let us consider α_0 and C_0 as in Proposition 5.30 and suppose (even if it means enlarging t_0) that $C_0 e^{-2e_\beta t_0} \leq \frac{\alpha_0}{2}$. We define for all n such that $S_n > t_0$:

$$t_n^* := \inf \left\{ t \in [t_0, S_n],\ \forall\, \tau \in [t, S_n],\ \left\| U_n(\tau) - R_\beta(\tau) - Ae^{-e_\beta \tau} Y_{+,\beta}(\tau) \right\|_{H^1\times L^2} \leq \alpha_0 \right\}.$$

By (5.59) and by continuity in time of U_n, R_β, and $Y_{+,\beta}$, t_n^* is indeed well-defined and we necessarily have $t_0 \leq t_n^* < S_n$.

Since (5.61) implies (5.62), for all $t \in [t_n^*, S_n]$,

$$\begin{aligned} \left\| U_n(t) - R_\beta(t) - Ae^{-e_\beta t} Y_{+,\beta}(t) \right\|_{H^1\times L^2} &\leq C_0 e^{-2e_\beta t} \\ &\leq C_0 e^{-2e_\beta t_0} \\ &\leq \frac{\alpha_0}{2}. \end{aligned}$$

Let us assume for the sake of contradiction that $t_n^* > t_0$ for some n. Then, observing the preceding inequality, we obtain (again by continuity in time of U_n, R_β, and $Y_{+,\beta}$) the existence of $\tau_n > 0$ such that $t_n^* - \tau_n \geq t_0$ and for all $t \in [t_n^* - \tau_n, S_n]$,

$$\left\|U_n(t) - R_\beta(t) - Ae^{-e_\beta t} Y_{+,\beta}(t)\right\|_{H^1\times L^2} \leq \frac{3\alpha_0}{4} < \alpha_0.$$

This contradicts the definition of t_n^* as an infimum. Hence $t_n^* = t_0$ and (5.61) (and thus (5.62)) holds on $[t_0, S_n]$ for all n.
This achieves the proof of Proposition 5.29. □

The existence of u^A (and U^A), as stated in Theorem 5.6 is a consequence of Proposition 5.29 and the continuity of the flow of (NLKG) for the weak $H^1 \times L^2$ topology. We will not detail the construction of U^A considering that it is a sort of cut and paste of what was done in order to prove Proposition 5.12 in the context of multiple solitons.

Similarly, we do not repeat the arguments exposed at the beginning of section 5.2 which justify that the map $A \mapsto u^A$ is one-to-one. We devote the next subsection to the proof of Proposition 5.30.

5.4.2 Proof of Proposition 5.30

We assume that $U_n(t)$ is defined on some interval $[t_n^*, S_n]$ and satisfies (5.61). We want to show that (5.62) holds, provided that the parameters α_0 and t_0 are well chosen.

In this subsection again, for notation purposes and ease of reading, we sometimes omit the index n and also write $\mathrm{O}(G(t))$ in order to refer to a function g which a priori depends on n and such that there exists $C \geq 0$ (independent of n) such that for all n large and for all $t \in [t_n^*, S_n]$, $|g(t)| \leq C|G(t)|$.

Step 1: Set up of a modulation argument

Lemma 5.13. *For $t_0 \geq 0$ sufficiently large and $\alpha_0 > 0$ sufficiently small, there exists a unique $\mathscr{C}^1$ function $x : [t_n^*, S_n] \to \mathbb{R}$ such that if we set*

$$W_n(t) := U_n(t) - \tilde{R}_\beta(t) - Ae^{-e_\beta t}\tilde{Y}_{+,\beta}(t),$$

with $\tilde{R}_\beta(t) := R_\beta(t, \cdot - x(t))$ and $\tilde{Y}_{+,\beta}(t) := Y_{+,\beta}(t, \cdot - x(t))$, then for all $t \in [t_n^*, S_n]$,

$$\left\langle W_n(t), \partial_x \tilde{R}_\beta(t)\right\rangle = 0. \tag{5.63}$$

Moreover there exists $K_1 > 0$ such that for all $t \in [t_n^*, S_n]$,

$$\|W_n(t)\|_{H^1\times L^2} + |x(t)| \leq K_1\alpha_0, \tag{5.64}$$

$$|\dot{x}(t)| \leq K_1\left(\|W_n(t)\|_{H^1\times L^2} + e^{-2e_\beta t}\right). \tag{5.65}$$

Remark 5.31. Notice that by uniqueness of the function x and by definition of u_n (see (5.59)), we have $W_n(S_n) = 0$ and $x(S_n) = 0$.

Proof. The existence of x such that (5.63) is granted and the existence of $K_2 > 0$ such that

$$\|W_n(t)\|_{H^1\times L^2} + |x(t)| \leq K_2\alpha_0$$

are standard consequences of the implicit function theorem.
Now, let us prove (5.65). For this, we notice that $W = W_n$ satisfies the following equation:

$$\begin{aligned}\partial_t W = &\begin{pmatrix} 0 & Id \\ \partial_x^2 - Id + f'(\tilde{Q}_\beta) & 0 \end{pmatrix}\left(W + Ae^{-e_\beta t}\tilde{Y}_{+,\beta}\right) + \begin{pmatrix} 0 \\ f(u) - f(\tilde{Q}_\beta) - f'(\tilde{Q}_\beta)(u - \tilde{Q}_\beta)\end{pmatrix} \\ &+ \dot{x}(t)\partial_x\tilde{R}_\beta - Ae^{-e_\beta t}\left(\partial_t\tilde{Y}_{+,\beta} - \dot{x}(t)\partial_x\tilde{Y}_{+,\beta} - e_\beta\tilde{Y}_{+,\beta}\right),\end{aligned} \tag{5.66}$$

where $\tilde{Q}_\beta(t,x) = Q_\beta(t, x - x(t))$.

Since $\frac{d}{dt}\langle W, \partial_x\tilde{R}_\beta\rangle = 0$, we have:

$$\langle\partial_t W, \partial_x\tilde{R}_\beta\rangle + \langle W, \partial_{tx}\tilde{R}_\beta - \dot{x}\partial_x^2\tilde{R}_\beta\rangle = 0.$$

Observing moreover that

$$f(u) - f(\tilde{Q}_\beta) - f'(\tilde{Q}_\beta)(u - \tilde{Q}_\beta) = \mathrm{O}\left(\|U - \tilde{R}_\beta\|^2_{H^1\times L^2}\right) = \mathrm{O}\left(\|W(t)\|^2_{H^1\times L^2} + e^{-2e_\beta t}\right)$$

by Taylor formula (f is $\mathscr{C}^2$), we have thus:

$$\begin{aligned}0 =& \left\langle\begin{pmatrix} 0 & Id \\ \partial_x^2 - Id + f'(\tilde{Q}_\beta) & 0\end{pmatrix}W, \partial_x\tilde{R}_\beta\right\rangle + \mathrm{O}\left(\|W(t)\|^2_{H^1\times L^2} + e^{-2e_\beta t}\right) + \dot{x}(t)\|\partial_x R_\beta\|^2_{H^1\times L^2} \\ &+ Ae^{-e_\beta t}\left\langle(\beta + \dot{x})\partial_x\tilde{Y}_{+,\beta} + e_\beta\tilde{Y}_{+,\beta}, \partial_x\tilde{R}_\beta\right\rangle - (\beta + \dot{x})\langle W, \partial_x^2\tilde{R}_\beta\rangle \\ =& \,\dot{x}\left(\|\partial_x R_\beta\|^2_{H^1\times L^2} + Ae^{-e_\beta t}\langle\partial_x\tilde{Y}_{+,\beta}, \partial_x\tilde{R}_\beta\rangle - \langle W, \partial_x^2\tilde{R}_\beta\rangle\right) + \mathrm{O}\left(\|W(t)\|^2_{H^1\times L^2} + e^{-2e_\beta t}\right) \\ &+ \left\langle\begin{pmatrix} \beta\partial_x & Id \\ \partial_x^2 - Id + f'(\tilde{Q}_\beta) & \beta\partial_x\end{pmatrix}W, \partial_x\tilde{R}_\beta\right\rangle + Ae^{-e_\beta t}\left\langle\begin{pmatrix} \beta\partial_x & Id \\ \partial_x^2 - Id + f'(\tilde{Q}_\beta) & \beta\partial_x\end{pmatrix}\tilde{Y}_{+,\beta}, \partial_x\tilde{R}_\beta\right\rangle.\end{aligned}$$

Notice that we have used $\langle Y_{+,\beta}, \partial_x R_\beta\rangle = 0$ (see Proposition 5.9). We now observe that

$$\begin{pmatrix} \beta\partial_x & Id \\ \partial_x^2 - Id + f'(\tilde{Q}_\beta) & \beta\partial_x\end{pmatrix} = -J\tilde{H}_\beta,$$

where $\tilde{H}_\beta$ is the matrix operator defined like H_β by replacing Q_β by $\tilde{Q}_\beta$, that is

$$\tilde{H}_\beta := \begin{pmatrix} -\partial_x^2 + Id - f'(\tilde{Q}_\beta) & -\beta\partial_x \\ \beta\partial_x & Id\end{pmatrix}. \tag{5.67}$$

We obtain:

$$\left\langle\begin{pmatrix} \beta\partial_x & Id \\ \partial_x^2 - Id + f'(\tilde{Q}_\beta) & \beta\partial_x\end{pmatrix}\tilde{Y}_{+,\beta}, \partial_x\tilde{R}_\beta\right\rangle = \left\langle\tilde{Y}_{+,\beta}, \tilde{H}_\beta J\partial_x\tilde{R}_\beta\right\rangle = \left\langle\tilde{Y}_{+,\beta}, 0\right\rangle = 0.$$

Finally we can choose t large enough such that

$$Ae^{-e_\beta t}\left|\langle\partial_x\tilde{Y}_{+,\beta}, \partial_x\tilde{R}_\beta\rangle\right| \leq \frac{1}{4}\|\partial_x R_\beta\|^2_{H^1\times L^2}$$

and we can take $\alpha_0 > 0$ sufficiently small such that

$$|\langle W, \partial_x^2 \tilde{R}_\beta \rangle| \le K_2 \alpha_0 \|\partial_x^2 \tilde{R}_\beta\|_{H^1\times L^2} \le \frac{1}{4}\|\partial_x R_\beta\|^2_{H^1\times L^2}$$

and such that $K_2\alpha_0 \le 1$ (then $\|W(t)\|^2_{H^1\times L^2} \le \|W(t)\|_{H^1\times L^2}$). Consequently for t large and α_0 small, we have:

$$\begin{aligned}|\dot{x}(t)| &\le \frac{2}{\|\partial_x R_\beta\|^2_{H^1\times L^2}} \left|\left\langle W, \begin{pmatrix} \beta\partial_x & \partial_x^2 - Id + f'(\tilde{Q}_\beta) \\ Id & \beta\partial_x \end{pmatrix} \partial_x \tilde{R}_\beta \right\rangle\right| + \mathrm{O}\left(\|W(t)\|_{H^1\times L^2} + e^{-2e_\beta t}\right) \\ &\le K_3 \left(\|W(t)\|_{H^1\times L^2} + e^{-2e_\beta t}\right),\end{aligned}$$

for some constant $K_3 > 0$. Finally, take $K_1 := \max(K_2, K_3)$ to obtain Lemma 5.13. □

Step 2: Control of particular directions

Let us denote

$$\alpha_{\pm,\beta}(t) := \langle W(t), \tilde{Z}_{\pm,\beta}(t)\rangle, \tag{5.68}$$

where $\tilde{Z}_{\pm,\beta}(t) := Z_{\pm,\beta}(t, \cdot - x(t))$.

Lemma 5.14. *We have for all $t \in [t_n^*, S_n]$,*

$$\left|\frac{d\alpha_{+,\beta}}{dt}(t) - e_\beta \alpha_{+,\beta}(t)\right| + \left|\frac{d\alpha_{-,\beta}}{dt}(t) + e_\beta \alpha_{-,\beta}(t)\right| = \mathrm{O}\left(\|W(t)\|^2_{H^1\times L^2} + e^{-2e_\beta t}\right).$$

Proof. Observe that

$$\begin{aligned}\partial_t \tilde{Z}_{\pm,\beta}(t,x) &= -(\beta + \dot{x}(t))\partial_x Z_{\pm,\beta}(t, x - x(t)) \\ &= -(\beta + \dot{x}(t))\partial_x \tilde{Z}_{\pm,\beta}(t,x).\end{aligned}$$

Thus

$$\frac{d\alpha_{\pm,\beta}}{dt} = \langle \partial_t W, \tilde{Z}_{\pm,\beta}\rangle + \langle W, -\beta\partial_x \tilde{Z}_{\pm,\beta}\rangle + \langle W, -\dot{x}\partial_x \tilde{Z}_{\pm,\beta}\rangle. \tag{5.69}$$

By (5.65), we obtain

$$\langle W, -\dot{x}\partial_x \tilde{Z}_{\pm,\beta}\rangle = \mathrm{O}\left(\|W(t)\|^2_{H^1\times L^2} + e^{-2e_\beta t}\right). \tag{5.70}$$

In addition, defining $\tilde{\mathscr{H}}_\beta$ like $\mathscr{H}_\beta$ by replacing Q_β by $\tilde{Q}_\beta$, we have by (5.66)

$$\begin{aligned}\langle \partial_t W, \tilde{Z}_{\pm,\beta}\rangle + \langle W, -\beta\partial_x \tilde{Z}_{\pm,\beta}\rangle &= \langle W, \tilde{\mathscr{H}}_\beta \tilde{Z}_{\pm,\beta}\rangle + \dot{x}\langle \partial_x \tilde{R}_\beta, \tilde{Z}_{\pm,\beta}\rangle \\ &\quad + Ae^{-e_\beta t}\left\langle \tilde{Y}_{+,\beta}, \tilde{\mathscr{H}}_\beta \tilde{Z}_{\pm,\beta}\right\rangle + \mathrm{O}\left(\|W(t)\|^2_{H^1\times L^2} + e^{-2e_\beta t}\right) \\ &\quad + Ae^{-e_\beta t}\left(\dot{x}\langle \partial_x \tilde{Y}_{+,\beta}, \tilde{Z}_{\pm,\beta}\rangle + e_\beta \langle \tilde{Y}_{+,\beta}, \tilde{Z}_{\pm,\beta}\rangle\right).\end{aligned} \tag{5.71}$$

Since

$$\langle \tilde{Y}_{+,\beta}, \tilde{Z}_{\pm,\beta}\rangle = \langle Y_{+,\beta}, Z_{\pm,\beta}\rangle = \begin{cases} 1 & \text{if we take } \pm = - \\ 0 & \text{if we take } \pm = +, \end{cases}$$

we always obtain

$$Ae^{-e_\beta t}\left\langle \tilde{Y}_{+,\beta}, \tilde{\mathscr{H}}_\beta \tilde{Z}_{\pm,\beta}\right\rangle + Ae^{-e_\beta t} e_\beta \langle \tilde{Y}_{+,\beta}, \tilde{Z}_{\pm,\beta}\rangle = 0. \tag{5.72}$$

We have in addition

$$|e^{-e_\beta t}\dot{x}| \leq \frac{1}{2}\left(e^{-2e_\beta t} + \dot{x}^2\right) = \mathrm{O}\left(\|W(t)\|^2_{H^1\times L^2} + e^{-2e_\beta t}\right)$$

and $\langle \partial_x \tilde{R}_\beta, \tilde{Z}_{+,\beta}\rangle$ by Proposition 5.9.

Hence, gathering (5.69), (5.70), (5.71), and (5.72), we infer:

$$\frac{d\alpha_{\pm,\beta}}{dt} = \pm e_\beta \langle W, \tilde{Z}_{\pm,\beta}\rangle + \mathrm{O}\left(\|W(t)\|^2_{H^1\times L^2} + e^{-2e_\beta t}\right),$$

which proves Lemma 5.14. □

Step 3: Exponential control of $\|W_n\|_{H^1\times L^2}$

Let us introduce the functional

$$\mathcal{F}_W(t) := \langle \tilde{H}_\beta W(t), W(t)\rangle. \tag{5.73}$$

(We recall that $\tilde{H}_\beta$ is defined in (5.67).)

Lemma 5.15 (Control of $\mathcal{F}_W$)**.** *There exists $C > 0$ such that for t_0 sufficiently large, we have for all n such that $S_n \geq t_0$, for all $t \in [t_n^*, S_n]$:*

$$|\mathcal{F}_W(t)| \leq C\left(\|W(t)\|^3_{H^1\times L^2} + e^{-3e_\beta t} + e^{-e_\beta t}|\alpha_{+,\beta}(t)|\right).$$

Proof. We have

$$\begin{aligned}\mathcal{F}_W(t) &= \langle \tilde{H}_\beta(U-\tilde{R}_\beta), U-\tilde{R}_\beta\rangle - 2Ae^{-e_\beta t}\langle \tilde{H}_\beta \tilde{Y}_{+,\beta}, W\rangle \\ &= \langle \tilde{H}_\beta(U-\tilde{R}_\beta), U-\tilde{R}_\beta\rangle - 2Ae^{-e_\beta t}\alpha_{+,\beta}.\end{aligned} \tag{5.74}$$

Notice that we have used $\tilde{H}_\beta \tilde{Y}_{+,\beta} = \tilde{Z}_{+,\beta}$ and $\langle \tilde{Y}_{+,\beta}, \tilde{Z}_{+,\beta}\rangle = 0$.

Now let us focus on the quadratic term $\langle \tilde{H}_\beta(U-\tilde{R}_\beta), U-\tilde{R}_\beta\rangle$. This term rewrites

$$\langle \tilde{H}_\beta(U-\tilde{R}_\beta), U-\tilde{R}_\beta\rangle = \int_{\mathbb{R}} \left\{\varepsilon_2^2 + (\partial_x\varepsilon_1)^2 + \varepsilon_1^2 - f'(\tilde{Q}_\beta)\varepsilon_1^2 + 2\beta\varepsilon_2\partial_x\varepsilon_1\right\}(t,x)\,dx,$$

where ε_1 and ε_2 are defined as follows:

$$\varepsilon_1(t,x) := u(t,x) - Q_\beta(t, x - x(t)) \quad \text{and} \quad \varepsilon_2(t,x) := \partial_t u(t,x) - \partial_t Q_\beta(t, x - x(t)).$$

In a compact manner, we can write:

$$\varepsilon_1 := u - \tilde{Q}_\beta \quad \text{and} \quad \varepsilon_2 := \partial_t u + \beta\partial_x \tilde{Q}_\beta$$

since $\partial_t Q_\beta(t,x) = -\beta\partial_x Q_\beta(t,x)$.

We observe that

$$\int_{\mathbb{R}} \left\{(\partial_t Q_\beta(t,x-x(t)))^2 + (\partial_x Q_\beta(t,x-x(t)))^2 + (Q_\beta(t,x-x(t)))^2\right\}\,dx$$

$$= \int_{\mathbb{R}} \left\{ (\partial_t Q_\beta(t,x))^2 + (\partial_x Q_\beta(t,x))^2 + (Q_\beta(t,x))^2 \right\} \, dx$$

and

$$\int_{\mathbb{R}} \partial_x Q_\beta(t, x - x(t)) \partial_t Q_\beta(t, x - x(t)) \, dx = \int_{\mathbb{R}} \partial_x Q_\beta(t,x) \partial_t Q_\beta(t,x) \, dx.$$

Considering that u_n and Q_β are solutions of (NLKG), the energy and the momentum of these solutions as defined in introduction are conserved. Thus, there exists $C_n \in \mathbb{R}$ such that

$$\begin{aligned}
&\langle \tilde{H}_\beta(U - \tilde{R}_\beta), U - \tilde{R}_\beta \rangle(t) \\
&\quad = C_n + 2 \int_{\mathbb{R}} \left\{ F(u(t,x)) - F(\tilde{Q}_\beta(t,x)) - \frac{1}{2} f'(\tilde{Q}_\beta(t,x)) \varepsilon_1(t,x)^2 \right\} \, dx \\
&\quad - 2 \int_{\mathbb{R}} \partial_t u(t,x) \left(-\beta \partial_x \tilde{Q}_\beta(t,x) \right) \, dx - 2 \int_{\mathbb{R}} \partial_x u(t,x) \partial_x \tilde{Q}_\beta(t,x) \, dx - 2 \int_{\mathbb{R}} u(t,x) \tilde{Q}_\beta(t,x) \, dx \\
&\quad - 2\beta \int_{\mathbb{R}} \partial_t u(t,x) \partial_x Q_\beta(t, x - x(t)) \, dx - 2\beta \int_{\mathbb{R}} \partial_x u(t,x) \partial_t Q_\beta(t, x - x(t)) \, dx.
\end{aligned}$$

Now, we notice that

$$\int_{\mathbb{R}} \partial_x u(t,x) \partial_x \tilde{Q}_\beta(t,x) \, dx = - \int_{\mathbb{R}} u(t,x) \partial_x^2 \tilde{Q}_\beta(t,x) \, dx$$

and

$$\partial_x \tilde{Q}_\beta(t,x) = \partial_x Q_\beta(t, x - x(t)).$$

Thus

$$\begin{aligned}
&\langle \tilde{H}_\beta(U - \tilde{R}_\beta), U - \tilde{R}_\beta \rangle(t) \\
&\quad = C_n + 2 \int_{\mathbb{R}} \left\{ F(u) - F(\tilde{Q}_\beta) - f(\tilde{Q}_\beta)\varepsilon_1 - \frac{1}{2} f'(\tilde{Q}_\beta) \varepsilon_1^2 \right\} (t,x) \, dx \\
&\quad + 2 \int_{\mathbb{R}} u(t,x) \left((1-\beta^2) \partial_x^2 \tilde{Q}_\beta(t,x) - \tilde{Q}_\beta(t,x) + f(\tilde{Q}_\beta(t,x)) \right) \, dx.
\end{aligned} \tag{5.75}$$

The last integral is zero by the equation satisfied by Q_β. By means of Taylor inequality, we claim

$$\begin{aligned}
&\int_{\mathbb{R}} \left\{ F(u(t,x)) - F(\tilde{Q}_\beta(t,x)) - f(\tilde{Q}_\beta(t,x))\varepsilon_1(t,x) - \frac{1}{2} f'(\tilde{Q}_\beta(t,x)) \varepsilon_1(t,x)^2 \right\} \, dx \\
&\quad = \mathrm{O} \left(\|U - \tilde{R}_\beta(t)\|_{H^1 \times L^2}^3 \right) \\
&\quad = \mathrm{O} \left(\|W(t)\|_{H^1 \times L^2}^3 + e^{-3 e_\beta t} \right).
\end{aligned} \tag{5.76}$$

Hence, collecting (5.74), (5.75), and (5.76), we have:

$$\mathcal{F}_W(t) = C_n + \mathrm{O} \left(\|W(t)\|_{H^1 \times L^2}^3 + e^{-3 e_\beta t} + e^{-e_\beta t} |\alpha_{+,\beta}(t)| \right).$$

On the other hand, we immediately have $|\mathcal{F}_W(t)| \leq C \|W(t)\|_{H^1 \times L^2}^2$. Since $W(S_n) = 0$ and $\alpha_{+,\beta}(S_n) = 0$, we deduce that $|C_n| \leq C e^{-3 e_\beta S_n} \leq C e^{-3 e_\beta t}$ for all $t \in [t_n^*, S_n]$. □

Corollary 5.16. *There exists $C > 0$ such that for t_0 sufficently large, we have for all n such that $S_n \geq t_0$, for all $t \in [t_n^*, S_n]$:*

$$|\mathcal{F}_W(t)| \leq C \left(\|W(t)\|^3_{H^1\times L^2} + e^{-e_\beta t} \sup_{t'\in[t,S_n]} \|W(t')\|^2_{H^1\times L^2} + e^{-3e_\beta t} \right).$$

Proof. From Lemma 5.14 we deduce that

$$\forall\, t \in [t_n^*, S_n], \qquad \left| \frac{d}{dt}\left(e^{-e_\beta t}\alpha_{+,\beta}\right) \right| \leq C \left(e^{-e_\beta t} \|W\|^2_{H^1\times L^2} + e^{-3e_\beta t} \right).$$

Given that $e^{-e_\beta S_n}\alpha_{+,\beta}(S_n) = 0$, we deduce by integration of the preceding inequality that:

$$\forall\, t \in [t_n^*, S_n], \qquad \left| e^{-e_\beta t}\alpha_{+,\beta}(t) \right| \leq C \left(e^{-e_\beta t} \sup_{t'\in[t,S_n]} \|W(t')\|^2_{H^1\times L^2} + e^{-3e_\beta t} \right).$$

Now the corollary follows from the combination of this last result with Lemma 5.15. □

Lemma 5.17. *There exists $c > 0$ such that for $\alpha_0 > 0$ sufficiently small and $t_0 \geq 0$ sufficiently large, for all n such that $S_n \geq t_0$, for all $t \in [t_n^*, S_n]$:*

$$\begin{cases} \left| \frac{d}{dt}\alpha_{+,\beta}(t) - e_\beta\alpha_{+,\beta}(t) \right| \leq c \min\left\{ \left(\sup_{t'\in[t,S_n]} \left(\alpha^2_{+,\beta}(t') + \alpha^2_{-,\beta}(t')\right) + e^{-2e_\beta t} \right), \alpha^2_{+,\beta}(t) + \alpha^2_{-,\beta}(t) + e^{-e_\beta t} \right\} \\ \left| \frac{d}{dt}\alpha_{-,\beta}(t) + e_\beta\alpha_{-,\beta}(t) \right| \leq c \min\left\{ \left(\sup_{t'\in[t,S_n]} \left(\alpha^2_{+,\beta}(t') + \alpha^2_{-,\beta}(t')\right) + e^{-2e_\beta t} \right), \alpha^2_{+,\beta}(t) + \alpha^2_{-,\beta}(t) + e^{-e_\beta t} \right\}. \end{cases} \tag{5.77}$$

Proof. Due to Proposition 5.10, we have on the one hand

$$\|W\|^2_{H^1\times L^2} \leq \frac{1}{\mu}\mathcal{F}_W(t) + \frac{1}{\mu^2}\left(\alpha^2_{+,\beta} + \alpha^2_{-,\beta}\right). \tag{5.78}$$

On the other,

$$|\mathcal{F}_W(t)| \leq C \left(\|W\|^3_{H^1\times L^2} + e^{-e_\beta t} \sup_{t'\in[t,S_n]} \|W(t')\|^2_{H^1\times L^2} + e^{-3e_\beta t} \right). \tag{5.79}$$

Now, inserting (5.79) into (5.78), we obtain

$$\|W\|^2_{H^1\times L^2} \leq \frac{2C}{\mu}e^{-3e_\beta t} + \frac{2}{\mu^2} \sup_{t'\in[t,S_n]} \left(\alpha^2_{+,\beta}(t') + \alpha^2_{-,\beta}(t')\right) \tag{5.80}$$

provided α_0 is chosen small enough such that $\frac{C}{\mu}\sup_{t'\in[t,S_n]}\|W(t')\|_{H^1\times L^2} \leq \frac{1}{4}$ (for all t) and t_0 is chosen large enough such that $\frac{C}{\mu}e^{-e_\beta t_0} \leq \frac{1}{4}$.

Then, combining (5.80) and the estimates obtained in Lemma 5.14, we deduce:

$$\left| \frac{d\alpha_{\pm,\beta}}{dt} \mp e_\beta\alpha_{\pm,\beta} \right| \leq C \left(\|W\|^2_{H^1\times L^2} + e^{-2e_\beta t} \right)$$

$$\leq C \left(\sup_{t' \in [t, S_n]} \left(\alpha_{+,\beta}^2(t') + \alpha_{-,\beta}^2(t') \right) + e^{-2e_\beta t} \right).$$

In order to avoid the supremum in front of the expression $\alpha_{+,\beta}^2 + \alpha_{-,\beta}^2$ and hence to obtain the second way of estimating $\frac{d}{dt}\alpha_{\pm,\beta} \mp e_\beta \alpha_{\pm,\beta}$ which is described by Lemma 5.17, we can rewrite (5.79) more simply as follows

$$|\mathcal{F}_W(t)| \leq C \left(\|W\|_{H^1 \times L^2}^3 + e^{-e_\beta t} + e^{-3e_\beta t} \right).$$

Then the analog of (5.80) takes the following form:

$$\|W\|_{H^1 \times L^2}^2 \leq C e^{-e_\beta t} + \left(\alpha_{+,\beta}^2(t) + \alpha_{-,\beta}^2(t) \right);$$

the last arguments remain unchanged. However note that, in this case, the obtained estimates are at the cost of a worse exponential term. □

Proposition 5.32. *There exists $C > 0$ such that for all $\alpha_0 \geq 0$ sufficiently small, for all t_0 sufficiently large, for all n such that $S_n \geq t_0$, for all $t \in [t_n^*, S_n]$, the following inequalities hold:*

$$|\alpha_{-,\beta}(t)| + |\alpha_{+,\beta}(t)| \leq C e^{-2e_\beta t} \tag{5.81}$$

and

$$\|W(t)\|_{H^1 \times L^2} \leq C e^{-\frac{3}{2} e_\beta t}. \tag{5.82}$$

Proof. Let us start with the system (5.77) obtained in Lemma 5.17. We have in particular

$$\begin{cases} \left| \dfrac{d}{dt}\alpha_{+,\beta} - e_\beta \alpha_{+,\beta} \right| \leq c \left(\alpha_{+,\beta}^2(t) + \alpha_{-,\beta}^2(t) + e^{-e_\beta t} \right) \\ \left| \dfrac{d}{dt}\alpha_{-,\beta} + e_\beta \alpha_{-,\beta} \right| \leq c \left(\alpha_{+,\beta}^2(t) + \alpha_{-,\beta}^2(t) + e^{-e_\beta t} \right). \end{cases} \tag{5.83}$$

Taking some inspiration in [10, paragraph 4.4.2], we will first show the existence of $M \geq 0$ such that for all t_0 large enough, for all $t \in [t_n^*, S_n]$,

$$\left| \alpha_{+,\beta}(t) \right| \leq M \left(\alpha_{-,\beta}^2(t) + e^{-e_\beta t} \right). \tag{5.84}$$

In order to prove (5.84), let us consider, for some positive constants M and $\tilde{M}$, the function

$$h : t \mapsto \alpha_{+,\beta}(t) - M\alpha_{-,\beta}^2(t) - \tilde{M} e^{-e_\beta t}$$

and show that it is always negative on $[t_n^*, S_n]$, provided M and $\tilde{M}$ are well chosen.
We compute

$$h'(t) = \alpha_{+,\beta}'(t) - 2M\alpha_{-,\beta}'(t)\alpha_{-,\beta}(t) + \tilde{M} e_\beta e^{-e_\beta t}.$$

By (5.83), we obtain

$$\begin{aligned} h' \geq\ & e_\beta \alpha_{+,\beta} - c \left(\alpha_{+,\beta}^2 + \alpha_{-,\beta}^2 + e^{-e_\beta t} \right) \\ & - 2M \left(-e_\beta \alpha_{-,\beta}^2 + c|\alpha_{-,\beta}| \left(\alpha_{+,\beta}^2 + \alpha_{-,\beta}^2 + e^{-e_\beta t} \right) \right) + \tilde{M} e_\beta e^{-e_\beta t}. \end{aligned}$$

Replacing $\alpha_{+,\beta}$ by its expression in terms of h, $\alpha_{-,\beta}^2$, and $e^{-e_\beta t}$ and using that

$$\left(h + M\alpha_{-,\beta}^2 + \tilde{M}e^{-e_\beta t}\right)^2 \leq 2h^2 + 4M^2\alpha_{-,\beta}^4 + 4\tilde{M}^2 e^{-2e_\beta t}$$

lead to the following estimate of h':

$$\begin{aligned} h' &\geq e_\beta h - 2c\left(1 + 2M|\alpha_{-,\beta}|\right)h^2 \\ &\quad + \left(e_\beta \tilde{M} - c - 2Mc|\alpha_{-,\beta}| - 4\tilde{M}^2 c e^{-e_\beta t} - 8M\tilde{M}^2 c|\alpha_{-,\beta}|e^{-e_\beta t}\right)e^{-e_\beta t} \\ &\quad + \alpha_{-,\beta}^2\left(3Me_\beta - c - 2Mc|\alpha_{-,\beta}| - 4cM^2\alpha_{-,\beta}^2 - 8M^3 c|\alpha_{-,\beta}|\alpha_{-,\beta}^2\right). \end{aligned}$$

Now we choose $M := \frac{c}{e_\beta}$ and $\tilde{M} := \frac{2c}{e_\beta}$ so that the expressions

$$3Me_\beta - c - 2Mc|\alpha_{-,\beta}| - 4cM^2\alpha_{-,\beta}^2 - 8M^3 c|\alpha_{-,\beta}|\alpha_{-,\beta}^2$$

and

$$e_\beta \tilde{M} - c - 2Mc|\alpha_{-,\beta}| - 4\tilde{M}^2 c e^{-e_\beta t} - 8M\tilde{M}^2 c|\alpha_{-,\beta}|e^{-e_\beta t}$$

are positive for α_0 sufficiently small and t_0 large enough. Thus for such values of α_0 and t_0, there exists $c_M > 0$ such that for all $t \in [t_n^*, S_n]$,

$$h'(t) \geq e_\beta h(t) - c_M h(t)^2.$$

At this stage, we can deduce that for all t in $[t_n^*, S_n]$, $h(t) \leq 0$. Assume for the sake of contradiction that there exists $\tilde{t}_n^* \in [t_n^*, S_n]$ such that $h(\tilde{t}_n^*) > 0$. Then, we can define

$$T := \sup\{t \in [\tilde{t}_n^*, S_n],\ h(t) > 0\}.$$

We necessarily have $h(T) = 0$. Indeed, $h(T) < 0$ is excluded by continuity of h in T (on the left side) and by definition of the supremum; if $h(T) > 0$, then $T < S_n$ (since $h(S_n) = -\tilde{M}e^{-e_\beta S_n}$) which leads once more to a contradiction, using the continuity of h in T (on the right side) and the definition of T as a supremum. It follows that $h'(T) \geq 0$; thus h is non-decreasing in the neighborhood of T and in particular, $h(t) \leq 0$ on $[T - \eta, T]$ for some $\eta > 0$. This again contradicts the definition of T.
Hence, for all t in $[t_n^*, S_n]$, $h(t) \leq 0$. Given that $-\alpha_{+,\beta}$ satisfies the same differential system as $\alpha_{+,\beta}$, we finally obtain (5.84).

Now, we have for all $t \in [t_n^*, S_n]$,

$$\sup_{t' \in [t, S_n]} |\alpha_{+,\beta}(t')| \leq M\left(\sup_{t' \in [t, S_n]} \alpha_{-,\beta}^2(t') + e^{-e_\beta t}\right).$$

By using Lemma 5.17 and even if it means reducing α_0 and increasing t_0, we obtain that for all $t \in [t_n^*, S_n]$,

$$|\alpha_{-,\beta}'(t) + e_\beta \alpha_{-,\beta}(t)| \leq C\left(\sup_{t' \in [t, S_n]} \alpha_{-,\beta}^2(t') + e^{-2e_\beta t}\right) \leq \frac{e_\beta}{10}|\alpha_{-,\beta}(t)| + Ce^{-2e_\beta t}.$$

We claim that this implies the estimates in Proposition 5.32. The preceding inequality rewrites as follows:

$$\left|\frac{d}{dt}(e^{e_\beta t}\alpha_{-,\beta})(t)\right| \leq \frac{e_\beta}{10}e^{e_\beta t} \sup_{t'\in[t,S_n]} |\alpha_{-,\beta}(t')| + Ce^{-e_\beta t}.$$

For t belonging to $[t_n^*, S_n]$, we obtain by integration on $[t, S_n]$:

$$e^{e_\beta t}|\alpha_{-,\beta}(t)| \leq Ce^{-e_\beta t} + \frac{e_\beta}{10}\int_t^{S_n} e^{e_\beta s} \sup_{t'\in[s,S_n]} |\alpha_{-,\beta}(t')|\ ds.$$

Passing to the supremum and defining $y(t) := e^{e_\beta t}\sup_{t'\in[t,S_n]}|\alpha_{-,\beta}(t')|$ on $[t_n^*, S_n]$, this leads to

$$y(t) \leq Ce^{-e_\beta t} + \frac{e_\beta}{10}\int_t^{S_n} y(s)\ ds. \tag{5.85}$$

Now, a standard Grönwall argument allows us to see that $y(t) \leq Ce^{-e_\beta t}$, which precisely provides the expected estimate of the parameter $\alpha_{-,\beta}$ in Proposition 5.32. Then, the similar estimate of the parameter $\alpha_{+,\beta}$ follows from the integration of the inequality

$$|\alpha'_{+,\beta}(t) - e_\beta\alpha_{+,\beta}(t)| \leq Ce^{-2e_\beta t}$$

and finally (5.82) follows from (5.80) and (5.81).
For the reader's convenience, let us write explicitly the Grönwall argument. The function $\psi(t) := e^{\frac{e_\beta}{10}t}\int_t^{S_n} y(s)\ ds$ is $\mathscr{C}^1$ on $[t_n^*, S_n]$ and for all $t \in [t_n^*, S_n]$,

$$\psi'(t) = e^{\frac{e_\beta}{10}t}\left(-y(t) + \frac{e_\beta}{10}\int_t^{S_n} y(s)\ ds\right) \geq -Ce^{-\frac{9e_\beta}{10}t}$$

by (5.85). Observing that $\psi(S_n) = 0$, it follows that $\psi(t) \leq Ce^{-\frac{9e_\beta}{10}t}$. As a consequence,

$$\int_t^{S_n} y(s)\ ds \leq Ce^{-e_\beta t}$$

and thus, by (5.85) again,

$$\forall\, t \in [t_n^*, S_n], \qquad y(t) \leq Ce^{-e_\beta t}.$$

□

Step 4: Improvement of the exponential decay rate

The goal of this paragraph is to optimize the exponential decay rate in the estimate of $\|W\|_{H^1\times L^2}$. We actually prove

Proposition 5.33. *The following estimate holds for all $t \in [t_n^*, S_n]$:*

$$\|W(t)\|_{H^1\times L^2} \leq Ce^{-2e_\beta t}.$$

Proof. Let us consider this time the derivative of $\mathcal{F}_W$ (and not the functional $\mathcal{F}_W$ itself). From the definition (5.73) and the symmetric property of $\tilde{H}_\beta$ for $\langle\cdot,\cdot\rangle$, we immediately obtain:

$$\frac{d}{dt}\mathcal{F}_W(t) = 2\langle \tilde{H}_\beta W, \partial_t W\rangle + \beta\int_{\mathbb{R}} \partial_x \tilde{Q}_\beta f''(\tilde{Q}_\beta) w_1^2\, dx,$$

where w_1 (resp. w_2) is the first (resp. the second) component of W. Replacing $\partial_t W$ by its expression obtained in (5.66) and noticing that

$$\begin{pmatrix} \beta\partial_x & Id \\ \partial_x^2 - Id + f'(\tilde{Q}_\beta) & \beta\partial_x \end{pmatrix} \tilde{Y}_{+,\beta} = J\tilde{H}_\beta \tilde{Y}_{+,\beta} = J\tilde{Z}_{+,\beta},$$

we have:

$$\begin{aligned} &\langle \tilde{H}_\beta W, \partial_t W\rangle \\ &= \left\langle \tilde{H}_\beta W, \begin{pmatrix} 0 & Id \\ \partial_x^2 - Id + f'(\tilde{Q}_\beta) & 0 \end{pmatrix} W \right\rangle + \left\langle \tilde{H}_\beta W, \begin{pmatrix} 0 \\ f(u) - f(\tilde{Q}_\beta) - f'(\tilde{Q}_\beta)(u - \tilde{Q}_\beta) \end{pmatrix} \right\rangle \\ &\quad + \langle \tilde{H}_\beta W, J\tilde{Z}_{+,\beta}\rangle + e_\beta A e^{-e_\beta t}\langle W, \tilde{Z}_{+,\beta}\rangle + \dot{x}\langle \tilde{H}_\beta W, \partial_x \tilde{R}_\beta\rangle + Ae^{-e_\beta t}\dot{x}\langle W, \tilde{H}\partial_x \tilde{Y}_{+,\beta}\rangle. \end{aligned}$$

Let us analyze each term appearing in the preceding decomposition. Since $\tilde{H}_\beta$ is self-adjoint, we infer:

$$\langle \tilde{H}_\beta W, \partial_x \tilde{R}_\beta\rangle = \langle W, \tilde{H}_\beta \partial_x \tilde{R}_\beta\rangle = 0 \tag{5.86}$$

and

$$\langle \tilde{H}_\beta W, J\tilde{Z}_{+,\beta}\rangle = -Ae^{-e_\beta t}\langle W, \mathscr{H}_\beta \tilde{Z}_{+,\beta}\rangle = -e_\beta A e^{-e_\beta t}\langle W, \tilde{Z}_{+,\beta}\rangle. \tag{5.87}$$

Moreover,

$$\left|\left\langle \tilde{H}_\beta W, \begin{pmatrix} 0 \\ f(u) - f(\tilde{Q}_\beta) - f'(\tilde{Q}_\beta)(u - \tilde{Q}_\beta) \end{pmatrix} \right\rangle\right| \le C\|W\|_{H^1\times L^2}\left(\|W\|^2_{H^1\times L^2} + e^{-2e_\beta t}\right). \tag{5.88}$$

We notice that (5.65) implies

$$|Ae^{-e_\beta t}\dot{x}\langle W, \tilde{H}_\beta \partial_x \tilde{Y}_{+,\beta}\rangle| \le Ce^{-e_\beta t}\|W(t)\|^2_{H^1\times L^2} + Ce^{-3e_\beta t}\|W(t)\|_{H^1\times L^2}. \tag{5.89}$$

Defining $\tilde{T} := -\partial_x^2 + Id - f'(\tilde{Q}_\beta)$, it remains us to examine

$$\begin{aligned} &\left\langle \tilde{H}_\beta W, \begin{pmatrix} 0 & Id \\ -\tilde{T} & 0 \end{pmatrix} W \right\rangle \\ &\qquad = \left\langle \begin{pmatrix} \tilde{T} & 0 \\ 0 & Id \end{pmatrix} W, \begin{pmatrix} 0 & Id \\ -\tilde{T} & 0 \end{pmatrix} W \right\rangle + \left\langle \begin{pmatrix} 0 & -\beta\partial_x \\ \beta\partial_x & 0 \end{pmatrix} W, \begin{pmatrix} 0 & Id \\ -\tilde{T} & 0 \end{pmatrix} W \right\rangle. \end{aligned}$$

On the one hand, we observe that

$$\left\langle \begin{pmatrix} \tilde{T} & 0 \\ 0 & Id \end{pmatrix} W, \begin{pmatrix} 0 & Id \\ -\tilde{T} & 0 \end{pmatrix} W \right\rangle = 0$$

and on the other

$$\left\langle \begin{pmatrix} 0 & -\beta\partial_x \\ \beta\partial_x & 0 \end{pmatrix} W, \begin{pmatrix} 0 & Id \\ -\tilde{T} & 0 \end{pmatrix} W \right\rangle = \int_{\mathbb{R}} \left\{-\beta\partial_x w_2 w_2 + \beta\partial_x w_1(-\tilde{T}w_1)\right\}\, dx$$

$$= \beta \int_{\mathbb{R}} \partial_x w_1 f'(\tilde{Q}_\beta) w_1$$
$$= -\frac{\beta}{2} \int_{\mathbb{R}} \partial_x \tilde{Q}_\beta f''(\tilde{Q}_\beta) w_1^2 \, dx.$$

Gathering the above lines, we deduce

$$\left| \frac{d\mathcal{F}_W}{dt} \right| \leq C \left(\|W\|^3_{H^1 \times L^2} + e^{-2e_\beta t} \|W\|_{H^1 \times L^2} + e^{-e_\beta t} \|W\|^2_{H^1 \times L^2} \right).$$

Using the estimate of $\|W\|_{H^1 \times L^2}$ obtained in Proposition 5.32, we infer:

$$\left| \frac{d\mathcal{F}_W}{dt} \right| \leq C \left(e^{-2e_\beta t} \|W\|_{H^1 \times L^2} + e^{-e_\beta t} \|W\|^2_{H^1 \times L^2} \right). \tag{5.90}$$

By integration of (5.90), we obtain for all $t \in [t^*, S_n]$:

$$\begin{aligned} |\mathcal{F}_W(t)| &\leq C \sup_{\tau \in [t, S_n]} \|W(\tau)\|_{H^1 \times L^2} \int_t^{S_n} e^{-2e_\beta \tau} \, d\tau + C \sup_{\tau \in [t, S_n]} \|W(\tau)\|^2_{H^1 \times L^2} \int_t^{S_n} e^{-e_\beta \tau} \, d\tau \\ &\leq C \sup_{\tau \in [t, S_n]} \|W(\tau)\|_{H^1 \times L^2} e^{-2e_\beta t} + C \sup_{\tau \in [t, S_n]} \|W(\tau)\|^2_{H^1 \times L^2} e^{-e_\beta t}. \end{aligned} \tag{5.91}$$

At this stage, using once again the coercivity property provided by Proposition 5.10 and the estimates on $\alpha_{\pm,\beta}$ given in Proposition 5.32, and taking t_0 large enough so that $e^{-e_\beta t} < \frac{1}{2}$ for all $t \geq t_0$, we have:

$$\sup_{\tau \in [t, S_n]} \|W(\tau)\|^2_{H^1 \times L^2} \leq C \left(\sup_{\tau \in [t, S_n]} \|W(\tau)\|_{H^1 \times L^2} e^{-2e_\beta t} + e^{-4e_\beta t} \right).$$

We now deduce the existence of $C > 0$ such that for all n, for all $t \in [t_n^*, S_n]$,

$$\|W(t)\|_{H^1 \times L^2} \leq C e^{-2e_\beta t}.$$

□

Now, Proposition 5.30 is obtained as a corollary of Proposition 5.33. The triangular inequality implies

$$\|U_n(t) - R_\beta(t) - A e^{-e_\beta t} Y_{+,\beta}(t)\|_{H^1 \times L^2} \leq \|W_n(t)\|_{H^1 \times L^2} + C|x(t)|$$

and since $x(S_n) = 0$, the result follows from the integration of

$$|\dot{x}(t)| \leq C \left(\|W_n(t)\|_{H^1 \times L^2} + e^{-2e_\beta t} \right) \leq C e^{-2e_\beta t}.$$

5.5 Classification of the asymptotic soliton-like solutions

Let us consider a solution u of (NLKG), denote $U = \begin{pmatrix} u \\ \partial_t u \end{pmatrix}$, and assume that

$$\|U(t) - R_\beta(t)\|_{H^1 \times L^2} \to 0, \quad \text{as } t \to +\infty.$$

We want to show that U equals to U^A, for some $A \in \mathbb{R}$. In this section again, we consider the one-dimensional case.

5.5.1 Modulation of U and coercivity property

Lemma 5.18. *There exist $T \in \mathbb{R}$, $C > 0$, and $x : [T, +\infty) \to \mathbb{R}$ of class $\mathscr{C}^1$ such that, denoting*

$$\tilde{R}_\beta(t,x) := \begin{pmatrix} Q_\beta(t, x - x(t)) \\ \partial_t Q_\beta(t, x - x(t)) \end{pmatrix} \quad \text{and} \quad E := U - \tilde{R}_\beta,$$

we have $\|E(t)\|_{H^1 \times L^2} \to 0$ as $t \to +\infty$, $x(t) \to 0$ as $t \to +\infty$, and for all $t \geq T$,

$$\left\langle E(t), \partial_x \tilde{R}_\beta(t) \right\rangle = 0 \tag{5.92}$$

and

$$|\dot{x}(t)| \leq C \|E(t)\|_{H^1 \times L^2}. \tag{5.93}$$

Proof. Lemma 5.18 is proved similarly as Lemma 5.13. Defining

$$\tilde{Q}_\beta(t,x) := Q_\beta(t, x - x(t)),$$

the equation satisfied by E reads:

$$\begin{aligned} \partial_t E = &\begin{pmatrix} 0 & Id \\ \partial_x^2 - Id + f'(\tilde{Q}_\beta) & 0 \end{pmatrix} E + \dot{x} \partial_x \tilde{R}_\beta \\ &+ \begin{pmatrix} 0 \\ f(u) - f(\tilde{Q}_\beta) - f'(\tilde{Q}_\beta)(u - \tilde{Q}_\beta) \end{pmatrix}. \end{aligned} \tag{5.94}$$

□

Let us denote $\varepsilon_1(t,x) := u(t,x) - Q_\beta(t, x - x(t))$ and $\varepsilon_2(t,x) := \partial_t u(t,x) - \partial_t Q_\beta(t, x - x(t))$ so that $E(t,x) := \begin{pmatrix} \varepsilon_1(t,x) \\ \varepsilon_2(t,x) \end{pmatrix}$.
Then, introduce the functional $\mathcal{F}_E$ defined as follows: for all $t \geq T$,

$$\mathcal{F}_E(t) := \int_{\mathbb{R}} \left\{ \varepsilon_2^2 + (\partial_x \varepsilon_1)^2 + \varepsilon_1^2 - f'(\tilde{Q}_\beta)\varepsilon_1^2 + 2\beta \varepsilon_2 \partial_x \varepsilon_1 \right\}(t,x)\, dx. \tag{5.95}$$

With analogous notations as that employed in the previous section, we denote

$$\tilde{Z}_{\pm,\beta}(t,x) := Z_{\pm,\beta}(t, x - x(t)).$$

Proposition 5.34. *There exists $\mu > 0$ such that for all $t \geq T$:*

$$\mathcal{F}_E(t) \geq \mu \|E(t)\|_{H^1 \times L^2}^2 - \frac{1}{\mu} \left[\left\langle E(t), \tilde{Z}_{-,\beta}(t) \right\rangle^2 + \left\langle E(t), \tilde{Z}_{+,\beta}(t) \right\rangle^2 \right]. \tag{5.96}$$

Proof. We observe that $\mathcal{F}_E(t) = \left\langle \tilde{H}_\beta E(t), E(t) \right\rangle$ so that Proposition 5.34 follows from Proposition 5.10 and from (5.92). □

5.5.2 Exponential control of $\|E(t)\|_{H^1 \times L^2}$

We improve the control of $\|E(t)\|_{H^1 \times L^2}$, proceeding as in section 5.4.

Control of the functional $\mathcal{F}_E$

Proposition 5.35. *We have*

$$\mathcal{F}_E(t) = \mathrm{O}\left(\|E(t)\|^3_{H^1\times L^2}\right). \tag{5.97}$$

Proof. Let us take again the proof of Lemma 5.15 (replacing A in that proof by 0 here). We obtain in the same way the existence of $K \in \mathbb{R}$ such that

$$\mathcal{F}_E(t) = K + 2\int_{\mathbb{R}} \left\{F(u) - F(\tilde{Q}_\beta) - f(\tilde{Q}_\beta)\varepsilon_1 - \frac{1}{2}f'(\tilde{Q}_\beta)\varepsilon_1^2\right\}(t,x)\ dx$$

By Taylor inequality,

$$\int_{\mathbb{R}} \left\{F(u) - F(\tilde{Q}_\beta) - f(\tilde{Q}_\beta)\varepsilon_1 - \frac{1}{2}f'(\tilde{Q}_\beta)\varepsilon_1^2\right\}(t,x)\ dx = \mathrm{O}\left(\|E(t)\|^3_{H^1\times L^2}\right).$$

It follows that

$$\mathcal{F}_E(t) = K + \mathrm{O}\left(\|E(t)\|^3_{H^1\times L^2}\right).$$

On the other hand, $\mathcal{F}_E(t) = \mathrm{O}\left(\|E(t)\|^2_{H^1\times L^2}\right)$; thus $K = 0$ and Proposition 5.35 is proved. □

Control of the unstable directions

Let us denote

$$\alpha_{\pm,\beta}(t) := \left\langle E(t), \tilde{Z}_{\pm,\beta}(t)\right\rangle.$$

Lemma 5.19. *We have*

$$\left|\frac{d\alpha_{+,\beta}}{dt} - e_\beta\alpha_{+,\beta}\right| + \left|\frac{d\alpha_{-,\beta}}{dt} + e_\beta\alpha_{-,\beta}\right| = \mathrm{O}\left(\|E(t)\|^2_{H^1\times L^2}\right).$$

Proof. The proof follows the same lines as that of Lemma 5.14 (by taking $A = 0$). We compute

$$\begin{aligned}
\frac{d\alpha_{\pm,\beta}}{dt} &= \left\langle \partial_t E, \tilde{Z}_{\pm,\beta}\right\rangle + \left\langle E, \partial_t \tilde{Z}_{\pm,\beta}\right\rangle - \dot{x}\left\langle E(t), \partial_x \tilde{Z}_{\pm,\beta}\right\rangle \\
&= \left\langle E, \begin{pmatrix} 0 & Id \\ \partial_x^2 - Id + f'(Q_\beta) & 0 \end{pmatrix}\tilde{Z}_{\pm,\beta}\right\rangle + \mathrm{O}\left(\|E\|^2_{H^1\times L^2}\right) \\
&\quad + \dot{x}\left\langle \partial_x \tilde{R}_\beta, \tilde{Z}_{\pm,\beta}\right\rangle + \left\langle E, \begin{pmatrix} -\beta\partial_x & 0 \\ 0 & -\beta\partial_x \end{pmatrix}\tilde{Z}_{\pm,\beta}\right\rangle \\
&= \left\langle E, \mathscr{H}_\beta \tilde{Z}_{\pm,\beta}\right\rangle + \mathrm{O}\left(\|E\|^2_{H^1\times L^2}\right) \\
&= \pm\, e_\beta \alpha_{\pm,\beta} + \mathrm{O}\left(\|E\|^2_{H^1\times L^2}\right),
\end{aligned}$$

which precisely yields Lemma 5.19. □

Proposition 5.36. *There exists $C \geq 0$ such that for t large enough,*

$$|\alpha_{-,\beta}(t)| \leq Ce^{-e_\beta t} \quad \text{and} \quad |\alpha_{+,\beta}(t)| \leq Ce^{-2e_\beta t}. \tag{5.98}$$

Proof. For t large enough, we obtain as a consequence of Lemma 5.34 and Proposition 5.35:

$$\|E(t)\|^2_{H^1\times L^2} \leq C\left(\alpha^2_{+,\beta} + \alpha^2_{-,\beta}\right). \tag{5.99}$$

Now, we deduce from Lemma 5.19 the following differential system, for some constant $C \geq 0$ and for all t large enough:

$$\begin{cases} \left|\dfrac{d\alpha_{+,\beta}}{dt} - e_\beta \alpha_{+,\beta}\right| \leq C\left(\alpha^2_{+,\beta} + \alpha^2_{-,\beta}\right) \\ \left|\dfrac{d\alpha_{-,\beta}}{dt} + e_\beta \alpha_{-,\beta}\right| \leq C\left(\alpha^2_{+,\beta} + \alpha^2_{-,\beta}\right). \end{cases} \tag{5.100}$$

Then, the argument exposed in [10, paragraph 4.4.2] shows that

$$|\alpha_{+,\beta}(t)| \leq C\alpha^2_{-,\beta}(t)$$

and allows us to conclude to (5.98). □

Proposition 5.37. *There exists $C \geq 0$ such that for t large enough,*

$$\|E(t)\|_{H^1\times L^2} \leq Ce^{-e_\beta t}. \tag{5.101}$$

Proof. This is an immediate consequence of (5.98) and (5.99). □

Identification of U with U^A for some $A \in \mathbb{R}$

Lemma 5.20. *There exist $A \in \mathbb{R}$ and $C \geq 0$ such that for t sufficiently large,*

$$|e^{e_\beta t}\alpha_{-,\beta}(t) - A| \leq Ce^{-e_\beta t}. \tag{5.102}$$

Proof. We have from (5.100):

$$\left|\frac{d}{dt}\left(e^{e_\beta t}\alpha_{-,\beta}(t)\right)\right| \leq Ce^{-e_\beta t}. \tag{5.103}$$

Thus the derivative of $t \mapsto e^{e_\beta t}\alpha_{-,\beta}(t)$ is integrable in the neighborhood of $+\infty$. Hence, there exists $A \in \mathbb{R}$ such that $e^{e_\beta t}\alpha_{-,\beta}(t) \to A$ as $t \to +\infty$. We finally obtain Lemma 5.20 by integration of (5.103). □

Lemma 5.21. *There exists $x_\infty \in \mathbb{R}$ such that for t sufficiently large,*

$$|x(t) - x_\infty| \leq Ce^{-e_\beta t}. \tag{5.104}$$

Proof. As (5.102) was a consequence of (5.103), (5.104) is obtained by means of the estimate

$$|\dot{x}(t)| \leq C\|E(t)\|_{H^1\times L^2} \leq Ce^{-e_\beta t}.$$

□

Now let $V(t,x) := U(t,x) - U^A(t, x - x_\infty)$ where A and x_∞ are defined in Lemma 5.20 and Lemma 5.21 respectively.

Lemma 5.22. *There exists $C \geq 0$ such that for t large enough, $\|V(t)\|_{H^1\times L^2} \leq Ce^{-e_\beta t}$.*

Proof. Define $V^A(t) := U^A(t) - R_\beta(t) - Ae^{-e_\beta t}Y_{+,\beta}(t)$ and decompose $V(t,x)$ as follows:

$$V(t,x) = E(t,x) + \tilde{R}_\beta(t,x) - R_\beta(t,x-x_\infty) - Ae^{-e_\beta t}Y_{+,\beta}(t,x-x_\infty) - V^A(t,x-x_\infty). \quad (5.105)$$

Now, we have

$$\tilde{R}_\beta(t,x) - R_\beta(t,x-x_\infty) = (x(t)-x_\infty)\partial_x R_\beta(t,x-x_\infty) + \mathrm{O}\left(|x(t)-x_\infty|^2\right). \quad (5.106)$$

Hence, Lemma 5.22 is obtained as a consequence of Proposition 5.37, Lemma 5.21, and the estimate of $\|V^A(t)\|_{H^1\times L^2}$ given by (5.4), that is $\|V^A(t)\|_{H^1\times L^2} = \mathrm{O}\left(e^{-2e_\beta t}\right)$. □

Let us decompose $V(t,x)$ as follows:

$$V(t,x) = \alpha^A_{+,\beta}(t)Y_{-,\beta}(t,x-x_\infty) + \alpha^A_{-,\beta}(t)Y_{+,\beta}(t,x-x_\infty) + \lambda(t)\partial_x R_\beta(t,x) + V_\perp(t,x), \quad (5.107)$$

where $\alpha^A_{+,\beta}(t) = \langle V(t), Z_{+,\beta}(t,\cdot - x_\infty)\rangle$, $\alpha^A_{-,\beta}(t) = \langle V(t), Z_{-,\beta}(t,\cdot - x_\infty)\rangle$ and

$$\lambda(t) := \frac{1}{\|\partial_x R_\beta\|_{H^1\times L^2}}\langle V(t) - \alpha^A_{+,\beta}(t)Y_{-,\beta}(t,\cdot - x_\infty) - \alpha^A_{-,\beta}(t)Y_{+,\beta}(t,x-x_\infty), \partial_x R_\beta(t)\rangle.$$

Then, we have

$$\langle V_\perp(t), \partial_x R_\beta(t)\rangle = \langle V_\perp(t), Z_{+,\beta}(t,\cdot - x_\infty)\rangle = \langle V_\perp(t), Z_{-,\beta}(t,\cdot - x_\infty)\rangle = 0.$$

Thus, by Proposition 5.10, there exists $C \geq 0$ such that

$$\|V_\perp(t)\|^2_{H^1\times L^2} \leq C\left\langle H_\beta V_\perp(t), V_\perp(t)\right\rangle. \quad (5.108)$$

We claim

Lemma 5.23. *The following assertions hold:*

1. $\langle H_\beta V(t), V(t)\rangle = \langle H_\beta V_\perp(t), V_\perp(t)\rangle + 2\alpha^A_{+,\beta}(t)\alpha^A_{-,\beta}(t)$.
2. $\frac{d}{dt}\langle H_\beta V(t), V(t)\rangle = \mathrm{O}\left(e^{-e_\beta t}\|V(t)\|^2_{H^1\times L^2}\right)$.
3. $|\alpha^A_{+,\beta}(t)| \leq Ce^{-e_\beta t}\|V(t)\|_{H^1\times L^2}$ *and* $|\alpha^A_{-,\beta}(t)| \leq Ce^{-e_\beta t}\int_t^{+\infty}\|V(t')\|_{H^1\times L^2}\,dt'$
4. $\lambda'(t) = \mathrm{O}\left(e^{-e_\beta t}\|V(t)\|_{H^1\times L^2} + e^{-e_\beta t}\int_t^{+\infty}\|V(t')\|_{H^1\times L^2}\,dt' + \|V_\perp(t)\|_{H^1\times L^2}\right)$.

Proof. The first assertion in Lemma 5.23 is obtained by the decomposition of V in terms of $V_\perp$ (5.107) and by means of the following properties (see Proposition 5.9):

$$\langle Z_{\pm,\beta}, Y_{\pm,\beta}\rangle = 0, \quad \langle Z_{\pm,\beta}, Y_{\mp,\beta}\rangle = 1, \quad \text{and} \quad H_\beta\left(\partial_x R_\beta\right) = 0.$$

Let us now prove assertion (2). Let $v := u - u^A$; recall that $U = \begin{pmatrix} u \\ \partial_t u\end{pmatrix}$ and $U^A = \begin{pmatrix} u^A \\ \partial_t u^A\end{pmatrix}$.
We observe that

$$\langle H_\beta V(t), V(t)\rangle = \int_{\mathbb{R}}\left\{(\partial_t v)^2 + (\partial_x v)^2 + v^2 - f'(Q_\beta)v^2 + 2\beta\partial_t v\partial_x v\right\}(t,x)\,dx.$$

Using the fact that u and u^A satisfy (NLKG), we obtain

$$
\begin{aligned}
&\frac{d}{dt}\langle H_\beta V(t), V(t)\rangle \\
&= -2\int_{\mathbb{R}} \left[f(u) - f(u^A) - f'(Q_\beta)v\right]\partial_t v\ dx \\
&\quad -2\beta\int_{\mathbb{R}} \partial_x v\left(f(u) - f(u^A)\right)\ dx + \beta\int_{\mathbb{R}} \partial_x Q_\beta f''(Q_\beta)v^2\ dx \\
&= \mathrm{O}\left(\|V(t)\|^3_{H^1\times L^2}\right) - 2\beta\int_{\mathbb{R}} \partial_x v\left(vf'(u^A) + \mathrm{O}(v^2)\right)\ dx + \beta\int_{\mathbb{R}} \partial_x Q_\beta f''(Q_\beta)v^2\ dx \\
&= \mathrm{O}\left(\|V(t)\|^3_{H^1\times L^2}\right) - 2\beta\int_{\mathbb{R}} \partial_x v\left(vf'(Q_\beta) + \mathrm{O}(v^2 + e^{-e_\beta t}v)\right)\ dx + \beta\int_{\mathbb{R}} \partial_x Q_\beta f''(Q_\beta)v^2\ dx \\
&= \mathrm{O}\left(\|V(t)\|^3_{H^1\times L^2} + e^{-e_\beta t}\|V(t)\|^2_{H^1\times L^2}\right).
\end{aligned}
$$

By Lemma 5.22, assertion (2) is thus proved.
In order to prove (3) and (4), let us write

$$
\partial_t V = \begin{pmatrix} 0 & Id \\ \partial_x^2 - Id + f'(Q_\beta) & 0 \end{pmatrix} V + \begin{pmatrix} 0 \\ f(u) - f(u^A) \end{pmatrix}. \tag{5.109}
$$

Then,

$$
\begin{aligned}
\frac{d}{dt}\alpha^A_{\pm,\beta}(t) &= \langle \partial_t V, Z_{\pm,\beta}(t,\cdot - x_\infty)\rangle + \langle V, \partial_t Z_{\pm,\beta}(t,\cdot - x_\infty)\rangle \\
&= \left\langle V, \begin{pmatrix} -\beta\partial_x & \partial_x^2 - Id + f'(Q_\beta) \\ Id & -\beta\partial_x \end{pmatrix} Z_{\pm,\beta}(t,\cdot - x_\infty)\right\rangle + \mathrm{O}\left(e^{-e_\beta t}\|V(t)\|_{H^1\times L^2}\right) \\
&= \pm e_\beta \alpha^A_{\pm,\beta}(t) + \mathrm{O}\left(e^{-e_\beta t}\|V(t)\|_{H^1\times L^2}\right)
\end{aligned}
$$

Note that we have used that $\partial_t Z_{\pm,\beta}(t,\cdot - x_\infty) = -\beta\partial_x Z_{\pm,\beta}(t,\cdot - x_\infty)$. We then deduce, in a similar way as for Proposition 5.36:

$$
|\alpha^A_{+,\beta}(t)| \le Ce^{-e_\beta t}\|V(t)\|_{H^1\times L^2}
$$

because $t \mapsto e^{-2e_\beta t}\|V(t)\|_{H^1\times L^2}$ is integrable in $+\infty$ and

$$
\left|e^{-e_\beta t}\alpha^A_{+,\beta}(t)\right| \le e^{-e_\beta t}\|V(t)\|_{H^1\times L^2} \underset{t\to+\infty}{\longrightarrow} 0.
$$

In addition we deduce

$$
\left|\frac{d}{dt}\left(e^{e_\beta t}\alpha^A_{-,\beta}(t)\right)\right| \le C\|V(t)\|_{H^1\times L^2}.
$$

Arguing similarly as for $\alpha^A_{+,\beta}(t)$, we then obtain $\left|\alpha^A_{-,\beta}(t)\right| \le Ce^{-e_\beta t}\int_t^{+\infty}\|V(t')\|_{H^1\times L^2}\ dt'$. This is due to the fact that $t \mapsto \|V(t)\|_{H^1\times L^2}$ is integrable in $+\infty$ and the fact that, by (5.105), (5.106), and (5.102),

$$
\alpha^A_{-,\beta}(t) = \alpha_{-,\beta}(t) - Ae^{-e_\beta t}\langle Y_{+,\beta}(t,\cdot - x_\infty), Z_{-,\beta}(t,\cdot - x_\infty)\rangle + \mathrm{O}(e^{-2e_\beta t})
$$

$$= \mathrm{O}(e^{-2e_\beta t}),$$

which explains that $e^{e_\beta t}\alpha^A_{-,\beta}(t) \to 0$ as $t \to +\infty$.
It remains to prove (4). By definition of $\lambda(t)$ and the decomposition (5.107) of V,

$$\begin{aligned}
\lambda'(t) &= \frac{1}{\|\partial_x R_\beta\|^2_{H^1\times L^2}} \left\langle \partial_t V - \alpha^A_{+,\beta}\partial_t Y_{-,\beta}(t,\cdot - x_\infty) - \alpha^A_{-,\beta}\partial_t Y_{+,\beta}(t,\cdot - x_\infty), \partial_x R_\beta \right\rangle \\
&\quad - \frac{1}{\|\partial_x R_\beta\|^2_{H^1\times L^2}} \left\langle (\alpha^A_{+,\beta})' Y_{-,\beta}(t,\cdot - x_\infty) + (\alpha^A_{-,\beta})' Y_{+,\beta}(t,\cdot - x_\infty), \partial_x R_\beta \right\rangle \\
&\quad + \frac{1}{\|\partial_x R_\beta\|^2_{H^1\times L^2}} \langle V_\perp, \partial_{xt} R_\beta \rangle \\
&= \frac{1}{\|\partial_x R_\beta\|^2_{H^1\times L^2}} \left\langle \partial_t V - \alpha^A_{+,\beta}\partial_t Y_{-,\beta}(t,\cdot - x_\infty) - \alpha^A_{-,\beta}\partial_t Y_{+,\beta}(t,\cdot - x_\infty), \partial_x R_\beta \right\rangle \\
&\quad + \frac{1}{\|\partial_x R_\beta\|^2_{H^1\times L^2}} \langle V_\perp, \partial_{xt} R_\beta \rangle.
\end{aligned}$$

since $\langle Y_{\pm,\beta}, \partial_x R_\beta\rangle = 0$ (we refer to Proposition 5.9). Hence,

$$\begin{aligned}
\lambda'(t) &= \left\langle \begin{pmatrix} 0 & Id \\ \partial_x^2 - Id + f'(Q_\beta) & 0 \end{pmatrix} V, \partial_x R_\beta \right\rangle + \mathrm{O}\left(e^{-e_\beta t}\|V\|_{H^1\times L^2}\right) + \frac{1}{\|\partial_x R_\beta\|^2_{H^1\times L^2}} \langle V_\perp, \partial_{xt} R_\beta\rangle \\
&\quad + \frac{1}{\|\partial_x R_\beta\|^2_{H^1\times L^2}} \left\langle \beta\alpha^A_{+,\beta}\partial_x Y_{-,\beta}(t,\cdot - x_\infty) + \beta\alpha^A_{-,\beta}\partial_x Y_{+,\beta}(t,\cdot - x_\infty), \partial_x R_\beta \right\rangle.
\end{aligned}$$

Now, using that $\partial_x R_\beta = JZ_0$ and ${}^tJ \begin{pmatrix} \beta\partial_x & Id \\ \partial_x^2 - Id + f'(Q_\beta) & \beta\partial_x \end{pmatrix} = H_\beta$, we have:

$$\begin{aligned}
&\left\langle \begin{pmatrix} 0 & Id \\ \partial_x^2 - Id + f'(Q_\beta) & 0 \end{pmatrix} V, \partial_x R_\beta \right\rangle \\
&\quad = \left\langle \begin{pmatrix} \beta\partial_x & Id \\ \partial_x^2 - Id + f'(Q_\beta) & \beta\partial_x \end{pmatrix} V, JZ_0 \right\rangle - \left\langle \begin{pmatrix} \beta\partial_x & 0 \\ 0 & \beta\partial_x \end{pmatrix} V, \partial_x R_\beta \right\rangle \\
&\quad = \langle H_\beta V, Z_0\rangle - \frac{1}{\|\partial_x R_\beta\|^2_{H^1\times L^2}} \left\langle \beta\alpha^A_{+,\beta}\partial_x Y_{-,\beta}(t,\cdot - x_\infty) - \beta\alpha^A_{-,\beta}\partial_x Y_{+,\beta}(t,\cdot - x_\infty), \partial_x R_\beta \right\rangle \\
&\qquad + \frac{1}{\|\partial_x R_\beta\|^2_{H^1\times L^2}} \langle V_\perp, \beta\partial_x^2 R_\beta\rangle.
\end{aligned}$$

Finally, let us notice that

$$\begin{aligned}
\langle H_\beta V, Z_0\rangle &= \langle H_\beta V_\perp, Z_0\rangle + \mathrm{O}\left(\alpha^A_{+,\beta} + \alpha^A_{-,\beta}\right) \\
&= \langle V_\perp, H_\beta Z_0\rangle + \mathrm{O}\left(\alpha^A_{+,\beta} + \alpha^A_{-,\beta}\right) \\
&= \mathrm{O}\left(\alpha^A_{+,\beta} + \alpha^A_{-,\beta} + \|V_\perp\|_{H^1\times L^2}\right).
\end{aligned}$$

Hence,

$$\lambda'(t) = \mathrm{O}\left(e^{-e_\beta t}\|V(t)\|_{H^1\times L^2} + |\alpha^A_{+,\beta}| + |\alpha^A_{-,\beta}| + \|V_\perp\|_{H^1\times L^2}\right)$$

and assertion (4) indeed holds. □

It follows from (5.108) and (1), (2), and (3) in Lemma 5.23 that for t large,

$$\|V_\perp(t)\|^2_{H^1\times L^2} \leq C\int_t^{+\infty} e^{-e_\beta t'}\|V(t')\|^2_{H^1\times L^2}\,dt' + Ce^{-2e_\beta t}\|V(t)\|_{H^1\times L^2}\int_t^{+\infty}\|V(t')\|_{H^1\times L^2}\,dt'. \quad (5.110)$$

Since

$$\int_t^{+\infty} e^{-e_\beta t'}\|V(t')\|^2_{H^1\times L^2}\,dt' \leq e^{-e_\beta t}\sup_{t'\geq t}\|V(t')\|_{H^1\times L^2}\int_t^{+\infty}\|V(t')\|_{H^1\times L^2}\,dt',$$

we deduce that for t large,

$$\|V_\perp(t)\|_{H^1\times L^2} \leq Ce^{-\frac{1}{2}e_\beta t}\sup_{t'\geq t}\|V(t')\|^{\frac{1}{2}}_{H^1\times L^2}\left(\int_t^{+\infty}\|V(t')\|_{H^1\times L^2}\,dt'\right)^{\frac{1}{2}}. \quad (5.111)$$

Lemma 5.24 (Estimate of $\|V(t)\|_{H^1\times L^2}$ in terms of $\int_t^{+\infty}\|V(t')\|_{H^1\times L^2}\,dt'$ and $\|V_\perp(t)\|_{H^1\times L^2}$)**.** *We have the existence of $C \geq 0$ such that for t sufficiently large:*

$$\|V(t)\|_{H^1\times L^2} \leq Ce^{-e_\beta t}\int_t^{+\infty}\|V(t')\|_{H^1\times L^2}\,dt' + C\left(\|V_\perp(t)\|_{H^1\times L^2} + \int_t^{+\infty}\|V_\perp(t')\|_{H^1\times L^2}\,dt'\right). \quad (5.112)$$

Proof. Using the decomposition (5.107) of V, we have:

$$\|V(t)\|_{H^1\times L^2} \leq C\left(|\alpha^A_{+,\beta}(t)| + |\alpha^A_{-,\beta}(t)| + |\lambda(t)| + \|V_\perp(t)\|_{H^1\times L^2}\right).$$

By Lemma 5.23 which gives estimates of $|\alpha^A_{+,\beta}(t)|$, $|\alpha^A_{-,\beta}(t)|$, and $\lambda'(t)$ in terms of $\|V(t)\|_{H^1\times L^2}$ and $\|V_\perp(t)\|_{H^1\times L^2}$, we obtain:

$$\lambda(t) = \mathrm{O}\left(e^{-e_\beta t}\int_t^{+\infty}\|V(t')\|_{H^1\times L^2}\,dt' + \int_t^{+\infty}\|V_\perp(t')\|_{H^1\times L^2}\,dt'\right). \quad (5.113)$$

Thus

$$\|V(t)\|_{H^1\times L^2} \leq C\left(e^{-e_\beta t}\|V(t)\|_{H^1\times L^2} + e^{-e_\beta t}\int_t^{+\infty}\|V(t')\|_{H^1\times L^2}\,dt' + \|V_\perp(t)\|_{H^1\times L^2}\right) + C\int_t^{+\infty}\|V_\perp(t')\|_{H^1\times L^2}\,dt'.$$

Hence, even if it means taking larger values of t, this finishes proving Lemma 5.24. □

Lemma 5.25. *We have*

$$\int_t^{+\infty} \|V(t')\|_{H^1\times L^2}\, dt' \leq Ce^{-e_\beta t} \sup_{t'\geq t} \|V(t')\|_{H^1\times L^2} \tag{5.114}$$

Proof. From (5.111), it follows that

$$\begin{aligned}\int_t^{+\infty} \|V_\perp(t')\|_{H^1\times L^2}\, dt' &\leq C \sup_{t'\geq t} \|V(t')\|_{H^1\times L^2}^{\frac{1}{2}} \left(\int_t^{+\infty} \|V(t')\|_{H^1\times L^2}\, dt'\right)^{\frac{1}{2}} \int_t^{+\infty} e^{-\frac{1}{2}e_\beta t'}\, dt' \\ &\leq Ce^{-\frac{1}{2}e_\beta t} \sup_{t'\geq t} \|V(t')\|_{H^1\times L^2}^{\frac{1}{2}} \left(\int_t^{+\infty} \|V(t')\|_{H^1\times L^2}\, dt'\right)^{\frac{1}{2}}.\end{aligned} \tag{5.115}$$

Gathering Lemma 5.24, (5.111), and (5.115), we obtain:

$$\begin{aligned}\|V(t)\|_{H^1\times L^2} \leq Ce^{-e_\beta t} &\int_t^{+\infty} \|V(t')\|_{H^1\times L^2} dt' \\ &+ C\left(e^{-\frac{1}{2}e_\beta t} \sup_{t'\geq t} \|V(t')\|_{H^1\times L^2}^{\frac{1}{2}} \left(\int_t^{+\infty} \|V(t')\|_{H^1\times L^2}\, dt'\right)^{\frac{1}{2}}\right).\end{aligned}$$

Now integrating the preceding inequality (which is obviously possible) leads to:

$$\begin{aligned}\int_t^{+\infty} \|V(t')\|_{H^1\times L^2}\, dt' \leq C &\int_t^{+\infty} \|V(t')\|_{H^1\times L^2}\, dt'. \int_t^{+\infty} e^{-e_\beta t'} dt' \\ &+ C \sup_{t'\geq t} \|V(t')\|_{H^1\times L^2}^{\frac{1}{2}} \left(\int_t^{+\infty} \|V(t')\|_{H^1\times L^2}\, dt'\right)^{\frac{1}{2}} \int_t^{+\infty} e^{-\frac{1}{2}e_\beta t'} dt'.\end{aligned}$$

Or more simply

$$\begin{aligned}\int_t^{+\infty} \|V(t')\|_{H^1\times L^2}\, dt' \leq Ce^{-e_\beta t} &\int_t^{+\infty} \|V(t')\|_{H^1\times L^2}\, dt' \\ &+ Ce^{-\frac{1}{2}e_\beta t} \sup_{t'\geq t} \|V(t')\|_{H^1\times L^2}^{\frac{1}{2}} \left(\int_t^{+\infty} \|V(t')\|_{H^1\times L^2}\, dt'\right)^{\frac{1}{2}}.\end{aligned}$$

Even if it means considering larger values of t, we obtain:

$$\int_t^{+\infty} \|V(t')\|_{H^1\times L^2}\, dt' \leq Ce^{-\frac{1}{2}e_\beta t} \sup_{t'\geq t} \|V(t')\|_{H^1\times L^2}^{\frac{1}{2}} \left(\int_t^{+\infty} \|V(t')\|_{H^1\times L^2}\, dt'\right)^{\frac{1}{2}},$$

which immediately implies the expected result. □

Now, let us show

Proposition 5.38. *For t sufficiently large, $V(t) = 0$.*

Proof. Gathering Lemma 5.24 and Lemma 5.25, we infer that for t large:

$$\|V(t)\|_{H^1\times L^2} \leq C\left(\sup_{t'\geq t} \|V_\perp(t')\|_{H^1\times L^2} + \int_t^{+\infty} \|V_\perp(t')\|_{H^1\times L^2}\, dt'\right).$$

From (5.111) and (5.115), we deduce

$$\|V(t)\|_{H^1\times L^2} \leq Ce^{-\frac{1}{2}e_\beta t} \sup_{t'\geq t} \|V(t')\|^{\frac{1}{2}}_{H^1\times L^2} \left(\int_t^{+\infty} \|V(t')\|_{H^1\times L^2}\, dt' \right)^{\frac{1}{2}}$$

Now, it results from Lemma 5.25 again that

$$\|V(t)\|_{H^1\times L^2} \leq Ce^{-e_\beta t} \sup_{t'\geq t} \|V(t')\|_{H^1\times L^2}.$$

Thus we deduce that $\|V(t)\|_{H^1\times L^2} = 0$ for large values of t. □

Finally let us observe below that $x_\infty = 0$ so that $U = U^A$.

Proposition 5.39. *There exists $t_0 \geq 0$ such that for all $t \geq t_0$, $U(t) = U^A(t)$.*

Proof. On the one hand, we have $\|U^A(t,\cdot - x_\infty) - R_\beta(t,\cdot - x_\infty)\|_{H^1\times L^2} \leq Ce^{-e_\beta t}$. On the other hand, we have $\|U(t) - R_\beta(t)\|_{H^1\times L^2} \to 0$ as $t \to +\infty$. Since we have $U(t) = U^A(t,\cdot - x_\infty)$, it follows from the triangular inequality that

$$\|R_\beta(t) - R_\beta(t,\cdot - x_\infty)\|_{H^1\times L^2} \to 0 \quad \text{as } t \to +\infty.$$

Hence $x_\infty = 0$ by the following claim, which is a consequence of Taylor formula.

Claim 5.40. *There exist $h_0 > 0$, $\epsilon_0 > 0$, and $\delta > 0$ such that*

1. *if $|h| \leq h_0$, then $\delta h^2 \leq \|Q(\cdot + h) - Q\|^2_{H^1} \leq 4\delta h^2$;*
2. *if $|h| > h_0$, then $\|Q(\cdot + h) - Q\|^2_{H^1} > \epsilon_0$.*

□

5.6 Appendix

5.6.1 Extension of the proofs to higher dimensions

The main parts of the proofs remain obviously unchanged. Essentially three notable adaptations are to be made, passing from the one-dimensional case to higher dimensions.

In a first instance, one has to be careful about how establishing several estimates. Although all estimates we have proved in the previous sections (in dimension 1) are identical for general d, the way we establish them when $d \geq 2$ can be altered.
For example, we point out that it is no longer possible to use the Sobolev embedding $H^1 \hookrightarrow L^\infty$ when $d \geq 2$. Particularly, multi-solitons in dimension $d \geq 2$ do not necessarily take values in $L^\infty(\mathbb{R}^d)$ and in order to estimate a quantity like

$$f(u) - f(\varphi) - f'(\varphi)(u - \varphi)$$

(as for proving Claim 5.15), we would proceed as follows: by **(H'1)**, we deduce that $|f''(r)| = p(p-1)|r|^{p-2}$, thus applying Taylor formula, we have for fixed time $t \in \mathbb{R}$ and position $x \in \mathbb{R}^d$,

$$|f(u) - f(\varphi) - f'(\varphi)(u - \varphi)|\,(t,x) \leq \frac{|u - \varphi|^2(t,x)}{2} \sup_{r\in[\varphi(t,x),u(t,x)]} |f''(r)|$$

$$\leq C|u-\varphi|^2(t,x)\left(|\varphi|^{p-2}(t,x)+|u|^{p-2}(t,x)\right).$$

Now, for all $\psi \in L^\infty(\mathbb{R}^d)$ and for all $z \in H^1(\mathbb{R}^d)$, Hölder inequality yields

$$\left|\int_{\mathbb{R}^d} |\psi||u(t)-\varphi(t)|^2|z|^{p-2}\,dx\right| \leq C\|\psi\|_{L^\infty}\left(\int_{\mathbb{R}^d}|u(t)-\varphi(t)|^p\,dx\right)^{\frac{2}{p}}\left(\int_{\mathbb{R}^d}|z|^p\,dx\right)^{\frac{p-2}{p}}.$$

Finally, replacing z by $\varphi(t)$ and $u(t)-\varphi(t)$, we obtain an estimate of

$$\int_{\mathbb{R}^d}\psi\left(f(u)-f(\varphi)-f'(\varphi)(u-\varphi)\right)\,dx$$

in terms of $\|u-\varphi\|_{H^1}$, $\|u\|_{H^1}$, and $\|\varphi\|_{H^1}$ due to the Sobolev embedding $H^1(\mathbb{R}^d)\hookrightarrow L^p(\mathbb{R}^d)$.

Secondly, in view of Proposition 5.9, one has to take into account, for all $i=1,\dots,N$ the d directions which generate the kernel of the operator $\mathscr{H}_{\beta_i}$ when we practice modulation in dimension d. For instance, in Lemma 5.5, we would define E as follows:

$$E := Z-\sum_{i=1}^N a_i\cdot\nabla R_i-\sum_{i=1}^N b_iY_{+,i},$$

with $a_i(t)\in\mathbb{R}^d$ and $b_i(t)\in\mathbb{R}$ such that for all $i=1,\dots,N$ and for all $j=1,\dots,d$:

$$\left\langle E(t),\partial_{x_j}R_i(t)\right\rangle = 0 \tag{5.116}$$

$$\left\langle E(t),Z_{-,i}(t)\right\rangle = 0. \tag{5.117}$$

But each time this extension does not affect the sequel; in other words, the estimates we are supposed to obtain afterwards and their proofs are the same.

A third change to be done concerns the way we define the different Lyapunov functionals which are studied throughout the article. To deal with dimensions greater or equal than 2, we reduce the problem to the case of a one-dimensional variable. For instance, let us explain how to generalize Step 4 in subsection 5.2.2 to all dimensions. The subset

$$\mathcal{M} := \bigcup_{i\neq j}\left\{\ell\in\mathbb{R}^d|\ \ell\cdot(\beta_j-\beta_i)=0\right\}$$

of $\mathbb{R}^d$ is of zero Lebesgue measure. Hence, there exists $\ell\in\mathbb{R}^d$ such that for all $i\neq j$,

$$\ell\cdot(\beta_j-\beta_i)\neq 0.$$

In particular $\ell\neq 0$ and, even if it means considering $\frac{\ell}{|\ell|}$, we can assume that $|\ell|=1$, so that $\forall\ i=1,\dots,N$, $|\ell\cdot\beta_i|<1$. Now, defining $\tilde{\beta}_i := \ell\cdot\beta_i$, and even if it means changing the permutation η, we have

$$-1<\tilde{\beta}_{\eta(1)}<\tilde{\beta}_{\eta(2)}<\cdots<\tilde{\beta}_{\eta(N)}<1.$$

Then, the direction described by ℓ is to be favored: we consider the following cut-off functions:

$$\psi_k(t)=\psi\left(\frac{1}{\sqrt{t}}\left(\ell\cdot x-\frac{\tilde{\beta}_{\eta(k)}+\tilde{\beta}_{\eta(k+1)}}{2}-\ell\cdot\frac{x_{\eta(k)}+x_{\eta(k+1)}}{2}\right)\right).$$

At this stage, the definition of the functions ϕ_k in terms of the ψ_k is kept unchanged and the corresponding Lyapunov functional is to be written:

$$\mathcal{F}_W(t) = \sum_{k=1}^{K} \int_{\mathbb{R}^d} \left(w_1^2 + (\partial_x w_1)^2 + w_2^2 - f'(Q_{\eta(k)}) w_1^2 + 2\beta_{\eta(k)} \cdot \nabla w_1 w_2 \right) \phi_k \, dx.$$

5.6.2 Proof of Corollary 5.1

The proof is an immediate adaptation of that of Proposition 4.12 in [10].

Let $A > 0$ and denote $t_A := -\frac{\ln(A)}{e_\beta}$. In the sense of the $H^1 \times L^2$-norm, we have:

$$\begin{aligned} U^1(t + t_A, \cdot + \beta t_A) &= R_\beta(t + t_A, \cdot + \beta t_A) + e^{-e_\beta (t + t_A)} Y_{+,\beta}(t + t_A, \cdot + \beta t_A) + \mathrm{O}\left(e^{-2e_\beta t}\right) \\ &= R_\beta(t) + A e^{-e_\beta t} Y_{+,\beta}(t) + \mathrm{O}\left(e^{-2e_\beta t}\right). \end{aligned}$$

Then, $\|U^1(t + t_A, \cdot + \beta t_A) - R_\beta(t)\|_{H^1 \times L^2} \underset{t \to +\infty}{\longrightarrow} 0$ so that there exist $\tilde{A} \in \mathbb{R}$ and $t_0 = t_0(\tilde{A}) \in \mathbb{R}$ such that for all $t \geq t_0$,

$$U^{\tilde{A}}(t) = U^1(t + t_A, \cdot + \beta t_A).$$

But on the other hand,

$$U^{\tilde{A}}(t) = R_\beta(t) + \tilde{A} e^{-e_\beta t} Y_{+,\beta}(t) + \mathrm{O}\left(e^{-2e_\beta t}\right).$$

Hence,

$$(A - \tilde{A}) e^{-e_\beta t} Y_{+,\beta}(t) = \mathrm{O}\left(e^{-2e_\beta t}\right),$$

which implies $A = \tilde{A}$. Consequently, $U^A(t) = U^1(t + t_A, \cdot + \beta t_A)$.

If $A < 0$, we have just to repeat the above argument with $-A$ instead of A.

Lastly, let us identify U^0. Given that R_β is a solution of (NLKG) which satisfies (5.5), Theorem 5.2 provides the existence of $A \in \mathbb{R}$ and of $t_0 \in \mathbb{R}$ such that for all $t \geq t_0$, $U^A(t) = R_\beta(t)$.
Since U^A satisfies (5.4), we deduce that

$$\left\| A e^{-e_\beta t} Y_{+,\beta}(t) \right\|_{H^1 \times L^2} \leq C e^{-2e_\beta t}.$$

Thus $A = 0$ and $U^0 = R_\beta$ is defined for all $t \in \mathbb{R}$.

5.6.3 A result of analytic theory of differential equations

Lemma 5.26. *Let $t_0 \in \mathbb{R}$, $\mathcal{A} : [t_0, +\infty) \to \mathbb{R}$ be a $\mathscr{C}^1$ bounded function, and $\xi : [t_0, +\infty) \to \mathbb{R}^+$ be continuous and integrable.*
If, for some $\rho > 0$,

$$\forall\, t \geq t_0, \qquad |\mathcal{A}'(t) + \rho \mathcal{A}(t)| \leq \xi(t) \sup_{t' \geq t} |\mathcal{A}(t')|,$$

then there exists $c > 0$ such that

$$\forall\, t \geq t_0, \qquad |\mathcal{A}(t)| \leq c e^{-\rho t}.$$

Proof. Let us assume that

$$\forall\, t \geq t_0, \qquad |\mathcal{A}'(t) + \rho\mathcal{A}(t)| \leq \xi(t) \sup_{t' \geq t} |\mathcal{A}(t')|, \tag{5.118}$$

for some $\rho > 0$. Then for all $t \geq t_0$,

$$|(e^{\rho t}\mathcal{A})'(t)| \leq \xi(t) e^{\rho t} \sup_{t' \geq t} |\mathcal{A}(t')|.$$

Let us consider $t \geq t_0$. For $t' \geq t$, we obtain by integration

$$|e^{\rho t'}\mathcal{A}(t') - e^{\rho t}\mathcal{A}(t)| \leq \int_t^{t'} \xi(s) e^{\rho s} \sup_{u \geq s} |\mathcal{A}(u)|\ ds.$$

This implies that, for $t' \geq t$,

$$e^{\rho t'}|\mathcal{A}(t')| \leq e^{\rho t}|\mathcal{A}(t)| + \sup_{u \geq t} |\mathcal{A}(u)| e^{\rho t'} \int_t^{t'} \xi(s)\ ds.$$

From the preceding line, we deduce that for all $t' \geq t$,

$$|\mathcal{A}(t')| \leq |\mathcal{A}(t)| + \sup_{u \geq t} |\mathcal{A}(u)| \int_t^{+\infty} \xi(s)\ ds. \tag{5.119}$$

Now we consider $t_1 \geq t_0$ such that $\int_{t_1}^{+\infty} \xi(s)\ ds < \frac{1}{2}$ (which is indeed possible given that $\int_t^{+\infty} \xi(s)\ ds \to 0$ as $t \to +\infty$). By passing to the supremum on t' in (5.119), we obtain for all $t \geq t_1$,

$$\sup_{t' \geq t} |\mathcal{A}(t')| \leq 2|\mathcal{A}(t)|.$$

Consequently, assumption (5.118) becomes

$$\forall\, t \geq t_1, \qquad |\mathcal{A}'(t) + \rho\mathcal{A}(t)| \leq 2\xi(t)|\mathcal{A}(t)|. \tag{5.120}$$

Let us define $y(t) := e^{\rho t}|\mathcal{A}(t)|$. By integration of (5.120), we obtain

$$\forall\, t \geq t_1, \qquad y(t) \leq y(t_1) + \int_{t_1}^{t} 2\xi(s) y(s)\ ds. \tag{5.121}$$

By a standard Grönwall argument, we conclude to the existence of $C > 0$ such that for all $t \geq t_1, y(t) \leq C$, which implies the desired result. For the sake of completeness, let us explicit this argument.

We define $Y(t) := \exp\left(-\int_{t_1}^t 2\xi(s)\ ds\right) \int_{t_1}^t 2\xi(s) y(s)\ ds$ for $t \geq t_1$. The function Y is $\mathscr{C}^1$ on $[t_1, +\infty)$ and for all $t \geq t_1$,

$$\begin{aligned} Y'(t) &= 2\xi(t) \exp\left(-\int_{t_1}^t 2\xi(s)\ ds\right) \left[y(t) - \int_{t_1}^t 2\xi(s) y(s)\ ds\right] \\ &\leq 2y(t_1)\xi(t) \exp\left(-\int_{t_1}^t 2\xi(s)\ ds\right) \end{aligned}$$

by (5.121). Integrating the preceding inequality and observing that $Y(t_1) = 0$, we have

$$Y(t) \leq \int_{t_1}^{t} 2\xi(s) y(t_1) \exp\left(- \int_{t_1}^{s} 2\xi(u)\, du \right) ds.$$

We then infer

$$\int_{t_1}^{t} 2\xi(s) y(s)\, ds = \exp\left(\int_{t_1}^{t} 2\xi(s)\, ds \right) Y(t) \leq 2y(t_1) \int_{t_1}^{t} \xi(s) \exp\left(\int_{s}^{t} 2\xi(u)\, du \right) ds. \tag{5.122}$$

Lastly, we denote $\nu := \int_{t_1}^{+\infty} \xi(s)\, ds$; gathering (5.121) and (5.122), we obtain

$$\forall\, t \geq t_1, \qquad y(t) \leq y(t_1) + 2y(t_1) e^{2\nu} \nu.$$

This achieves the proof of Lemma 5.26. ◻

Bibliography

[1] Miguel A. Alejo and Claudio Muñoz. Nonlinear stability of mKdV breathers. *Comm. Partial Differ. Equ.*, 324:233–262, 2013.

[2] Henri Berestycki and Pierre-Louis Lions. Non linear scalar field equations, I. Existence of a ground state. *Arch. Ration. Mech. Anal.*, 82(4):313–345, 1983.

[3] Jerry L. Bona, Panagiotis E. Souganidis, and Walter A. Strauss. Stability and instability of solitary waves of Korteweg-de Vries type. *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences*, 411(1841):395–412, 1987.

[4] Thomas Brabec and Ferenc Krausz. Intense few-cycle laser fields: Frontiers of nonlinear optics. *Rev. Mod. Phys.*, 72:545–591, 2000.

[5] Thierry Cazenave and Pierre-Louis Lions. Orbital stability of standing waves for some nonlinear Schrödinger equations. *Comm. Math. Phys.*, 85:549–561, 1982.

[6] Thierry Cazenave and Fred B. Weissler. The Cauchy problem for the nonlinear Schrödinger equation in H^1. *Manuscripta mathematica*, 61:477–494, 1988.

[7] Thierry Cazenave and Fred B. Weissler. The Cauchy problem for the critical nonlinear Schrödinger equation in H^s. *Nonlinear Analysis: Theory, Methods and Applications*, 14(10):807 – 836, 1990.

[8] Gong Chen and Jiaqi Liu. Soliton resolution for the modified KdV equation, 2019.

[9] Amy Cohen. Existence and regularity for solutions of the Korteweg-de Vries equation. *Arch. Ration. Mech. Anal.*, 71:143–175, 1979.

[10] Vianney Combet. Construction and characterization of solutions converging to solitons for supercritical gKdV equations. *Differential Integral Equations*, 23(5/6):513–568, 05 2010.

[11] Vianney Combet. Multi-soliton solutions for the supercritical gKdV equations. *Comm. Partial Differ. Equ.*, 36(3):380–419, 2011.

[12] Vianney Combet. Multi-existence of multi-solitons for the supercritical nonlinear Schrödinger equation in one dimension. *Discrete Contin. Dyn. Syst.*, 34(5):1961–1993, 2014.

[13] Raphaël Côte. Construction of solutions to the subcritical gKdV equations with a given asymptotical behavior. *Journal of Functional Analysis*, 241:143–211, 12 2006.

[14] Raphaël Côte. Large data wave operator for the generalized Korteweg-de Vries equations. *Differential and Integral Equations*, 19, 02 2006.

[15] Raphaël Côte. Construction of solutions to the L^2 -critical KdV equation with a given asymptotic behaviour. *Duke Math. J.*, 138(3):487–531, 06 2007.

[16] Raphaël Côte and Xavier Friederich. On smoothness and uniqueness of multi-solitons of the non-linear Schrödinger equations. *Communications in Partial Differential Equations*, 2021.

[17] Raphaël Côte and Stefan Le Coz. High-speed excited multi-solitons in nonlinear Schrödinger equations. *J. Math. P. Appl.*, 96:135–166, 2011.

[18] Raphaël Côte and Yvan Martel. Multi-travelling waves for the nonlinear Klein-Gordon equation. *Trans. Am. Math. Soc.*, 370(10):7461–7487, 2018.

[19] Raphaël Côte, Yvan Martel, and Frank Merle. Construction of multi-soliton solutions for the L^2-supercritical gKdV and NLS equations. *Rev. Mat. Iberoam.*, 27(1):273–302, 2011.

[20] Raphaël Côte and Claudio Muñoz. Multi-solitons for nonlinear Klein–Gordon equations. *Forum of Math., Sigma*, 2:e15, 2014.

[21] Wei Dai, Weihua Yang, and Daomin Cao. Continuous dependence of Cauchy problem for nonlinear Schrödinger equation in H^s. *J. of Differ. Equ.*, 255(7):2018–2064, 2013.

[22] Kanchan Kumar De, Amit Goyala, C.N. Kumar, and Amarendra K. Sarma. Few-cycle optical solitary waves in cascaded-quadratic–cubic–quintic nonlinear media. *Comm. Nonlinear Sci. Numer. Simul.*, 20(3):629–633, 2015.

[23] G. H. Derrick. Comments on nonlinear wave equations as models for elementary particles. *Journal of Mathematical Physics*, 5(9):1252–1254, 1964.

[24] Thomas Duyckaerts and Frank Merle. Dynamics of threshold solutions for energy-critical wave equation. *Int. Math. Res. Pap.*, 2008(4), 2008. rpn002.

[25] Thomas Duyckaerts and Frank Merle. Dynamic of threshold solutions for energy-critical NLS. *Geometric and Functional Analysis*, 18:1787–1840, 2009.

[26] Thomas Duyckaerts and Svetlana Roudenko. Threshold solutions for the focusing 3D cubic Schrödinger equation. *Rev. Mat. Iberoam.*, 26(1):1–56, 2010.

[27] Juan Dávila, Manuel del Pino, and Ignacio Guerra. Non-uniqueness of positive ground states of non-linear Schrödinger equations. *Proc. Lond. Math. Soc., Third Series*, 2:318–344, 2012.

[28] Wiktor Eckhaus and Peter C. Schuur. The emergence of solutions for the generalized Korteweg-de Vries equation from arbitrary initial conditions. *Math. Meth. Appl. Sci.*, 5:97–116, 1983.

[29] Domenico Felice. *A study of a nonlinear Schrödinger equation for optical fibers*. PhD thesis, Università degli Studi di Firenze, Firenze, IT, 2012.

[30] Xavier Friederich. Non dispersive solutions of the generalized Korteweg-de Vries equations are typically multi-solitons. *Annales de l'Institut Henri Poincaré C, Analyse non linéaire*, 2021.

[31] Xavier Friederich. On existence and uniqueness of asymptotic N-soliton-like solutions of the nonlinear Klein-Gordon equation. working paper or preprint, June 2021.

[32] Clifford S. Gardner, John M. Greene, Martin D. Kruskal, and Robert M. Miura. Korteweg-de Vries equation and generalizations. VI. Methods for exact solutions. *Communications on pure and applied mathematics*, 27(1):97–133, 1974.

[33] Basilis Gidas, Wei Ming Ni, and Louis Nirenberg. Symmetry and related properties via the maximum principle. *Comm. Math. Phys.*, 68(3):209–2432, 1979.

[34] Jean Ginibre and Giorgio Velo. On a class of nonlinear Schrödinger equations. I. The Cauchy problem, general case. *J. Funct. Anal.*, 32:1–32, 1979.

[35] Jean Ginibre and Giorgio Velo. The global Cauchy problem for the nonlinear Klein-Gordon equation. *Mathematische Zeitschrift*, 189(4):487–505, 1985.

[36] Manoussos Grillakis. Analysis of the linearization around a critical point of an infinite dimensional hamiltonian system. *Comm. Pure Appl. Math.*, 43(3):299–333, 1990.

[37] Manoussos Grillakis, Jalal Shatah, and Walter Strauss. Stability theory of solitary waves in the presence of symmetry, I. *J. Funct. Anal.*, 74:160–197, 1987.

[38] Manoussos Grillakis, Jalal Shatah, and Walter Strauss. Stability theory of solitary waves in the presence of symmetry, II. *J. Funct. Anal.*, 94:308–348, 1990.

[39] Ryogo Hirota. Exact solution of the Korteweg-de Vries equation for multiple collisions of solitons. *Phys. Rev. Lett.*, 27:1192–1194, Nov 1971.

[40] Pedro Isaza, Felipe Linares, and Gustavo Ponce. On decay properties of solutions of the k-generalized KdV equation. *Communications in Mathematical Physics*, 324(7):129–146, 2013.

[41] Pedro Isaza, Felipe Linares, and Gustavo Ponce. On the propagation of regularity and decay of solutions to the k-generalized Korteweg-de Vries equation. *Communications in Partial Differential Equations*, 40(7):1336–1364, 2015.

[42] Jaeduck Jang. Uniqueness of positive radial solutions of $\Delta u + f(u) = 0$ in $\mathbb{R}^N$, $N \geq 2$. *Nonlinear Anal.*, 73:2189–2198, 2010.

[43] Jacek Jendrej. Dynamics of strongly interacting unstable two-solitons for generalized Korteweg-de Vries equations, 2018, arXiv.

[44] Jacek Jendrej. Construction of two-bubble solutions for energy-critical wave equations. *Amer. J. Math.*, 141(1):55–118, 2019.

[45] Jacek Jendrej and Andrew Lawrie. Uniqueness of two-bubble wave maps, 2020, arxiv.

[46] Jacek Jendrej and Yvan Martel. Construction of multi-bubble solutions for the energy-critical wave equation in dimension 5. *J. Math. Pures Appl.*, 139:317–355, 2020.

[47] Tosio Kato. On the Cauchy problem for the (generalized) Korteweg-de Vries equation. *Studies in Applied Mathematics Adv. Math. Supplementary Studies*, 18:93–128, 1983.

[48] Tosio Kato. On nonlinear Schrödinger equations. II, H^s solutions and unconditional well-posedness. *J. of Anal. Math.*, 67:281–306, 1995.

[49] P. L. Kelley. Self-Focusing of Optical Beams. *Phys. Rev. Lett.*, 15:1005–1008, Dec 1965.

[50] Carlos E. Kenig, Gustavo Ponce, and Luis Vega. Well-posedness and scattering results for the generalized Korteweg-de Vries equation via the contraction principle. *Comm. Pure Appl. Math.*, 46:527–620, 1993.

[51] Rowan Killip, Tadahiro Oh, Oana Pocovnicu, and Monica Visan. Solitons and scattering for the cubic-quintic nonlinear Schrödinger equation on $\mathbb{R}^3$. *Archive for Rational Mechanics and Analysis*, 225:469–548, 2017.

[52] Joachim Krieger, Yvan Martel, and Pierre Raphaël. Two-soliton solutions to the three-dimensional gravitational Hartree equation. *Comm. Pure Appl. Math.*, 62:1501–1550, 2009.

[53] Joachim Krieger, Kenji Nakanishi, and Wilhelm Schlag. Global dynamics above the ground state energy for the one-dimensional NLKG equation. *Mathematische Zeitschrift*, 272, 11 2010.

[54] Kwok-Kun Kwong. Uniqueness of positive solutions of $\Delta u - u + u^p = 0$ in $\mathbb{R}^n$. *Arch. Ration. Mech. Anal.*, 105(3):243–266, 1989.

[55] G.L. Lamb. *Elements of Soliton Theory*. John Wiley & Sons, New York, 1980.

[56] Yang Lan. Blow-up solutions for L^2-supercritical gKdV equations with exactly k blow-up points. *Nonlinearity*, 30, 02 2016.

[57] Céline Laurent and Yvan Martel. Smoothness and exponential decay of L^2-compact solutions of the generalized KdV equations. *Comm. Partial Differ. Equ.*, 29:157–171, 2005.

[58] Peter D. Lax. Integrals of nonlinear equations of evolution and solitary waves. *Communications on pure and applied mathematics*, 21:467–490, 1968.

[59] Stefan Le Coz, Dong Li, and Tai-Peng Tsai. Fast-moving finite and infinite trains of solitons for nonlinear Schrödinger equations. *Proceedings of the Royal Society of Edinburgh: Section A Mathematics*, 145(6):1251–1282, 2015.

[60] Stefan Le Coz and Tai-Peng Tsai. Infinite soliton and kink-soliton trains for nonlinear Schrödinger equations. *Nonlinearity*, 27(11):2689–2709, oct 2014.

[61] Boris Malomed. *Nonlinear Schrödinger Equations*. Scott Alwyn, Encyclopedia of Nonlinear Science, New York, NY: Routledge, 2005.

[62] Mihai Maris. Existence of nonstationary bubbles in higher dimension. *J. Math. Pures Appl.*, 81:1207–1239, 2002.

[63] Yvan Martel. Asymptotic N-soliton-like solutions of the subcritical and critical generalized Korteweg-de Vries equations. *American Journal of Mathematics*, 127(5):1103–1140, 2005.

[64] Yvan Martel. Linear problems related to asymptotic stability of solitons of the generalized kdv equations. *SIAM Journal on Mathematical Analysis*, 38(3):759–781, 2006.

[65] Yvan Martel and Frank Merle. A Liouville theorem for the critical generalized Korteweg-de Vries equation. *J. Math. Pures Appl.*, 79:339–425, 2000.

[66] Yvan Martel and Frank Merle. Asymptotic stability of solitons for subcritical gKdV equations. *Arch. Ration. Mech. Anal.*, 157:219–254, 2001.

[67] Yvan Martel and Frank Merle. Instability of solitons for the critical generalized Korteweg-de Vries equation. *Geometric And Funct. Analysis*, 11(1):74–123, 2001.

[68] Yvan Martel and Frank Merle. Blow up in finite time and dynamics of blow up solutions for the L^2-critical generalized KdV equation. *Journal of the American Mathematical Society*, 15, 01 2002.

[69] Yvan Martel and Frank Merle. Nonexistence of blow-up solution with minimal L^2-mass for the critical gKdV equation. *Duke Math. J.*, 115(2):385–408, 11 2002.

[70] Yvan Martel and Frank Merle. Asymptotic stability of solitons of the subcritical gKdV equations revisited. *Nonlinearity*, 18:55–80, 2005.

[71] Yvan Martel and Frank Merle. Multisolitary waves for nonlinear Schrödinger equations. *Annales de l'Institut Henri Poincaré, C Analyse non linéaire*, 23:849–864, 2006.

[72] Yvan Martel and Frank Merle. Asymptotic stability of solitons of the gKdV equations with general nonlinearity. *Math. Ann.*, 341:391–427, 2008.

[73] Yvan Martel and Frank Merle. Description of two soliton collision for the quartic gKdV equation. *Ann. Math.*, 174(2):757–857, 2011.

[74] Yvan Martel and Frank Merle. Inelastic interaction of nearly equal solitons for the quartic gKdV equation. *Invent. Math.*, 183(3):563–648, 2011.

[75] Yvan Martel and Frank Merle. On the nonexistence of pure multi-solitons for the quartic gKdV equation. *Int. Math. Res. Not. IMRN*, (3):688–739, 2015.

[76] Yvan Martel and Frank Merle. Construction of multi-Solitons for the energy-critical wave equation in dimension 5. *Archive for Rational Mechanics and Analysis*, 222(3):1113 – 1160, 2016.

[77] Yvan Martel, Frank Merle, and Pierre Raphaël. Blow up for the critical generalized Korteweg–de Vries equation. I: Dynamics near the soliton. *Acta Math.*, 212(1):59–140, 2014.

[78] Yvan Martel, Frank Merle, and Pierre Raphaël. Blow up for the critical generalized Korteweg–de Vries equation. II: Minimal mass dynamics. *Journal of the European Mathematical Society*, 17(8):1855 – 1925, 2015.

[79] Yvan Martel, Frank Merle, and Pierre Raphaël. Blow up for the critical generalized Korteweg–de Vries equation. III: Exotic regimes. *Annali della Scuola Normale Superiore di Pisa*, XIV:575–631, 2015.

[80] Yvan Martel, Frank Merle, and Tai-Peng Tsai. Stability and asymptotic stability in the energy space of the sum of N solitons for subcritical gKdV equations. *Comm. Math. Phys*, 231:347–373, 2002.

[81] Yvan Martel, Frank Merle, and Tai-Peng Tsai. Stability in H^1 of the sum of K solitary waves for some nonlinear Schrödinger equations. *Duke Math J.*, 133(3):405–466, 2006.

[82] Yvan Martel and Tien Vinh Nguyen. Construction of 2-solitons with logarithmic distance for the one-dimensional cubic Schrödinger system. *Discrete & Continuous Dynamical Systems*, 40(3):1595–1620, 2020.

[83] Yvan Martel and Pierre Raphaël. Strongly interacting blow up bubbles for the mass critical nonlinear Schrödinger equation. *Ann. Sci. Ec. Norm. Supér. (4)*, 51(3):701–737, 2018.

[84] Kevin Mc Leod and James Serrin. Uniqueness of positive radial solutions of $\Delta u + f(u) = 0$ in $\mathbb{R}^n$. *Arch. Ration. Mech. Anal.*, 99:115–145, 1987.

[85] Frank Merle. Construction of solutions with exactly k blow-up points for the Schrödinger equation with critical nonlinearity. *Comm. Math. Phys.*, 129:223–240, 1990.

[86] Frank Merle. On uniqueness and continuation properties after blow-up time of self-similar solutions of nonlinear Schrödinger equation with critical exponent and critical mass. *Communications on pure and applied mathematics*, 15:203–254, 1992.

[87] Frank Merle. Determination of blow-up solutions with minimal mass for nonlinear Schrödinger equations with critical power. *Duke Math. J.*, 69(2):427–454, 02 1993.

[88] Mei Ming, Frederic Rousset, and Nikolay Tzvetkov. Multi-solitons and related solutions for the water-waves system. *SIAM J. Math. Anal.*, 47:897–954, 2015.

[89] Robert M. Miura. The Korteweg-de Vries equation: A survey of results. *SIAM Review*, 18(3):412–459, 1976.

[90] Claudio Muñoz. On the inelastic 2-soliton collision for gKdV equations with general nonlinearity. *Int. Math. Research Notices*, 9:1624–1719, 2010.

[91] Makoto Nakamura and Tohru Ozawa. Nonlinear Schrödinger equations in the Sobolev space of critical order. *Journal of Functional Analysis*, 155:364–380, 06 1998.

[92] Makoto Nakamura and Tohru Ozawa. The Cauchy problem for nonlinear Klein-Gordon equations in the Sobolev spaces. *Publ. Res. Inst. Math. Sci.*, 37, 11 2001.

[93] K. Nakanishi and W. Schlag. Global dynamics above the ground state energy for the focusing nonlinear Klein-Gordon equation. *Journal of Differential Equations*, 250(5):2299 – 2333, 2011.

[94] Tien Vinh Nguyen. Strongly interacting multi-solitons with logarithmic relative distance for the gKdV equation. *Nonlinearity*, 30(12):4614–4648, nov 2017.

[95] Tien Vinh Nguyen. Existence of multi-solitary waves with logarithmic relative distances for the NLS equation. *C. R. Math. Acad. Sci. Paris*, 357(1):13–58, 2019.

[96] Robert L. Pego and Michael I. Weinstein. Eigenvalues, and instabilities of solitary waves. *Philosophical Transactions of the Royal Society of London. Series A: Physical and Engineering Sciences*, 340:47–94, 1992.

[97] Robert L. Pego and Michael I. Weinstein. Asymptotic stability of solitary waves. *Communications in Mathematical Physics*, 164:305–349, 1994.

[98] Lev P. Pitaevskii and Sandro Stringari. *Bose-Einstein Condensation*. Oxford Science Publications, Oxford, UK: Oxford University Press. Clarendon Press., 2003.

[99] Stanislas I. Pohozaev. Eigenfunctions of the equation $\Delta u + \lambda f(u) = 0$. *Sov. Math. J.*, 5:1408–1411, 1965.

[100] Wilhelm Schlag. Spectral theory and nonlinear partial differential equations: A survey. *Discrete and Continuous Dynamical Systems*, 15, 07 2006.

[101] Peter C. Schuur. *Asymptotic Analysis of Solitons Problems*. Berlin: Springer-Verlag, 1986.

[102] Alexander Semenov. On the existence and uniqueness of multi-breathers of (mKdV). *preprint*.

[103] James Serrin and Moxun Tang. Uniqueness of ground states for quasilinear elliptic equations. *Indiana Univ. Math. J.*, 49:897–923, 2000.

[104] Terence Tao. Low regularity semi-linear wave equations. *Communications in Partial Differential Equations*, 24(3-4):599–629, 1999.

[105] Yoshio Tsutsumi. L^2-solutions for nonlinear Schrödinger equations and nonlinear groups. *Funkcial. Ekvac.*, 30(1):115–125, 1987.

[106] Frédéric Valet. *Etude de comportements en temps long de solutions des équations de Zakharov-Kuznetsov*. PhD thesis, Université de Strasbourg, Strasbourg, FR, 2020.

[107] Frédéric Valet. Asymptotic K-soliton-like solutions of the Zakharov-Kuznetsov type equations, 2021.

[108] Michael I. Weinstein. Modulational stability of ground states of nonlinear dispersive Schrödinger equations. *SIAM J. Math. Anal.*, 16:472–491, 1985.

[109] Michael I. Weinstein. Lyapunov stability of ground states of nonlinear dispersive evolution equations. *Comm. Pure Appl. Math.*, 39:51–67, 1986.

[110] Xu Yuan. On multi-solitons for the energy-critical wave equation in dimension 5. *Nonlinearity*, 32(12):5017–5048, nov 2019.

[111] Xu Yuan. Construction of excited multi-solitons for the 5D energy-critical wave equation. 2020.

[112] N. J. Zabusky and M. D. Kruskal. Interaction of "solitons" in a collisionless plasma and the recurrence of initial states. *Phys. Rev. Lett.*, 15:240–243, Aug 1965.

[113] Vladimir E. Zakharov and Aleksei B. Shabat. Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media. *Sov. Phys. JETP*, 34:62–69, 1972.